A comparative analysis of aggregation rules for coherent lower previsions

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Summary

We consider the problem of belief aggregating information when the information is expressed by means of coherent lower previsions.

We put together a number of rationality criteria and aggregation rules studied in different papers.

Specifically, we consider five aggregation rules, twelve rationality criteria and provide a detailed analysis of the properties satisfied by each rule.

Coherent lower previsions

Given a possibility space $\mathcal{X}$, a gamble is a bounded function $f : \mathcal{X} \to \mathbb{R}$.

Given the set $\mathcal{L}(\mathcal{X})$ a lower prevision is a function $\underline{P} : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$. Its conjugate upper prevision is given by $\overline{P}(f) = -\underline{P}(-f)\ \forall f$.

A lower prevision is coherent when

COH1. $\underline{P}(f) \geq \inf f$ for any f.

COH2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for any $f \in \mathcal{L}(\mathcal{X})$ and $\lambda > 0$.

COH3. $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ for any f, g.

If $\underline{P}$ is coherent and coincides with $\overline{P}$, it is called a linear prevision. Its restriction to events is a finitely additive probability.

In particular, we can use lower previsions to represent non-additive measures, sets of probability measures or preference relations.

Sensitivity analysis interpreation

A lower prevision determines a closed and convex set of linear previsions:

$$\mathcal{M}(\underline{P}) = \{P \text{ linear prevision} : P(f) \geq \underline{P}(f)\ \forall f \in \mathcal{L}(\mathcal{X})\}.$$

$$\underline{P} \text{ is coherent} \iff \underline{P}(f) = \min_{P \in \mathcal{M}(\underline{P})} P(f)\ \forall f.$$

When $\mathcal{M}(\underline{P}) \neq \emptyset$, we say that $\underline{P}$ avoids sure loss.

In addition, $\underline{P}(f)$ can be given a behavioural interpretation as the supremum acceptable buying price for f, meaning that $f - \underline{P}(f) + \varepsilon$ is a desirable transaction for all $\varepsilon > 0$.

We denote by $\underline{\mathbb{P}}(\mathcal{X})$ the set of lower previsions on $\mathcal{X}$, and by $\underline{\mathbb{P}}'(\mathcal{X})$ the subset of coherent lower previsions.

Aggregation rules

Assume we have a group of n experts, and that each of them models her uncertainty about an experiment in terms of a coherent lower prevision $\underline{P}_i$ on $\mathcal{L}(\mathcal{X})$.

An aggregation rule on coherent lower previsions is a map $A : \left(\underline{\mathbb{P}}'(\mathcal{X})\right)^n \to \underline{\mathbb{P}}(\mathcal{X})$.

We shall denote $\underline{P} := A(\underline{P}_1, \ldots, \underline{P}_n)$. This lower prevision summarises the opinions of the group.

Let us meet our contestants

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Which one is better?

Does anybody care?

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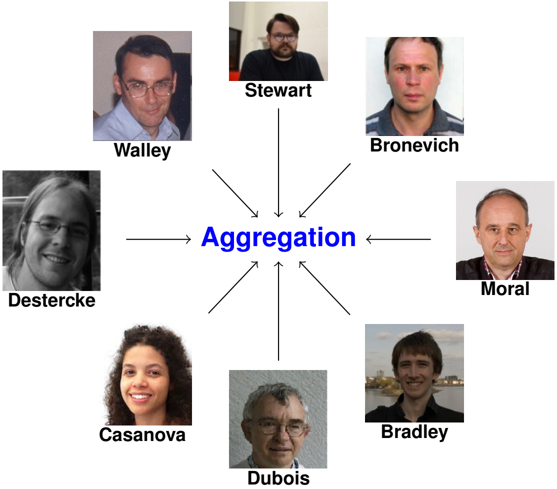

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Zero preservation If $\max_{i=1,\dots,n} \underline{P}_i(A) = 0$, then $\underline{P}(A) = 0$.

Properties satisfied by each rule

Property \ Rule	Conjunction	Disjunction	Mixture	Pareto	CD
Coherence	✗*	✓	✓	✗	✓
Unanimity	✓	✓	✓	✓	✓
Reconciliation	✗*	✓	✗	✓	✓
Indeterminacy	✓	✓	✓	✓	✓
Strong indeterminacy	✗	✓	✓	✓	✗
Strong Pareto	✓	✗	✗	✓	✗
Conjunction	✓	✗	✗	✓	✓
Total reconciliation	✗	✓	✗	✗	✗
Symmetry	✓	✓	✗	✓	✓
Monotonicity	✓	✓	✓	✓	✓
Idempotence	✓	✓	✓	✓	✓
Zero preservation	✗	✓	✓	✓	✗

(*) It holds if $\max_{i=1,\dots,n} \underline{P}_i$ avoids sure loss.

Conclusions

- The Pareto rule does not preserve coherence.
- The mixture rule is neither too precise nor too imprecise, but lacks some interesting properties.
- The CD rule circunvents some of the problems of the conjunction rule.

Future work:

- Other IP representations.
- Other aggregation rules.
- Dealing with conflict.

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Thank you for the attention...

...and for the questions!