On the closure of aggregation rules for imprecise probabilities

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Who are we?

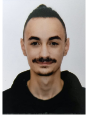
IMAGINE

E. Miranda

I.Montes

Who are we?


M.Iturrate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J.Salamanca

Summary

Let $\mathcal{X}$ be a finite space. Assume we have a group of n experts, and that each of them models her uncertainty about an experiment in terms of a coherent lower prevision $\underline{P}_i$ on $\mathcal{L}(\mathcal{X})$, with associated credal set $\mathcal{M}(\underline{P}_i)$.

An aggregation rule on coherent lower previsions is a map $\mathcal{A}$ that transforms this input into a lower prevision $\underline{P} := \mathcal{A}(\underline{P}_1, \ldots, \underline{P}_n)$ that summarises the opinions of the group.

We consider four aggregation rules and six different families, and study:

(a) if the aggregation rule is closed for a given family;

(b) if not, if there are particular cases when it is closed.

Let us meet our contestants

Conjunction ($\mathcal{A}_C$) It is the lower envelope of $\cap_{i=1}^{n}\mathcal{M}(\underline{P}_i)$, or, equivalently, the natural extension of $\max\left\{\underline{P}_1, \ldots, \underline{P}_n\right\}$.

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Disjunction ($\mathcal{A}_D$) $\mathcal{A}_D(f) = \min\left\{\underline{P}_1(f), \ldots, \underline{P}_n(f)\right\} \, \forall f$.

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Mixture ($\mathcal{A}_M$) Given $\alpha_1,\ldots,\alpha_n\geq 0$ with $\sum_{i=1}^{n}\alpha_i=1$, $\mathcal{A}_M(f)=\sum_{j=1}^{n}\alpha_j\underline{P}_j(f)\ \forall f$.

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Pareto ($\mathcal{A}_P$) $\mathcal{A}_P(f) = \min\left\{\max_{j=1,\ldots,n} \underline{P}_j(f), \min_{j=1,\ldots,n} \overline{P}_j(f)\right\} \; \forall f$.

Life is a war


AC
AD
AM
AP


2monot
belief
RE
COMPARATIVE
pbox
probint
Minitive

Comparative probabilities

A comparative probability model on singletons is determined by some $\mathcal{L} \subseteq \mathcal{X} \times \mathcal{X}$, so that $\mathcal{M}(\mathcal{L}) = \{P : P(\{x_i\}) \geq P(\{x_j\})\ \forall (x_i, x_j) \in \mathcal{L}\}$.

- $\mathcal{A}_C$ is closed for comparative probabilities. ✓✓
- $\mathcal{A}_D$ is not closed for comparative probabilities, but $\mathcal{A}_D(\underline{P}_1, \underline{P}_2)$ has a unique undominated outer approximation. ✓
- $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$ is a comparative probability iff $\mathcal{L}_1 = \mathcal{L}_2$. ✓
- $\mathcal{A}_P$ is closed for comparative probabilities. ✓✓

2-monotone capacities

$\underline{P}$ is 2-monotone when $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$ for any $A, B \subseteq \mathcal{X}$.

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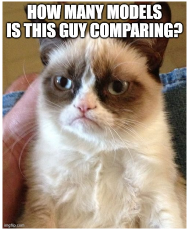
- $\mathcal{A}_C$ does not preserve 2-monotonicity, but it does so when $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex. ✓
- $\mathcal{A}_D$ does not preserve 2-monotonicity, but it does when either $\underline{P}_1$ or $\underline{P}_2$ is a categorical belief function. ✓
- $\mathcal{A}_M$ preserves 2-monotonicity. ✓✓
- $\mathcal{A}_P$ does not preserve 2-monotonicity. ✗

A lower probability $\underline{P}$ is a probability interval when $\mathcal{M}(\underline{P})$ is determined by the lower and upper probabilities of the singletons.



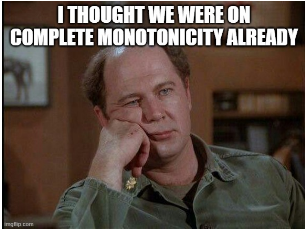
Probability intervals

A lower probability $\underline{P}$ is a probability interval when $\mathcal{M}(\underline{P})$ is determined by the lower and upper probabilities of the singletons.

- (De Campos et al. 1994) $\mathcal{A}_C$ preserves being a probability interval. ✓✓
- (De Campos et al. 1994) $\mathcal{A}_D$ does not preserve being a probability interval, but there is a unique outer approximation of $\mathcal{A}_D(\underline{P}_1, \underline{P}_2)$. ✓
- $\mathcal{A}_M$ does not preserve being a probability interval, but there is a unique outer approximation of $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$. ✓
- $\mathcal{A}_P$ does not preserve being a probability interval, but it does when $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent. ✓

Belief functions

$\underline{P}$ is a belief function (=completely monotone) when its möbius inverse is non-negative.



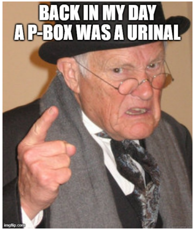
Belief functions

$\underline{P}$ is a belief function (=completely monotone) when its möbius inverse is non-negative.

- $\mathcal{A}_C$ does not preserve complete monotonicity. ✗
- $\mathcal{A}_D$ does not preserve complete monotonicity, but it does when either $\underline{P}_1$ or $\underline{P}_2$ is a degenerate probability measure. ✓
- $\mathcal{A}_M$ preserves complete monotonicity. ✓✓
- $\mathcal{A}_P$ does not preserve complete monotonicity. ✗

p-boxes

A lower probability $\underline{P}$ is a p-box when $\mathcal{M}(\underline{P})$ is uniquely determined by the lower and upper probabilities of the cumulative events.



p-boxes

A lower probability $\underline{P}$ is a p-box when $\mathcal{M}(\underline{P})$ is uniquely determined by the lower and upper probabilities of the cumulative events.

- The conjunction $\mathcal{A}_C$ of two p-boxes is a p-box iff $\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2) \neq \emptyset$. ✓✓
- $\mathcal{A}_D$ is not closed in the family of p-boxes, but $\mathcal{A}_D(\underline{P}_1, \underline{P}_2)$ has a unique undominated outer approximation in the family. ✓
- $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$ is a p-box iff $\mathcal{F}_1 \cup \mathcal{F}_2$ is totally ordered by set inclusion, where $\mathcal{F}_1, \mathcal{F}_2$ denote the focal elements of $\underline{P}_1, \underline{P}_2$. ✓
- $\mathcal{A}_P$ is not closed in the family of p-boxes. ✗

Minitive measures

A lower probability is minitive iff $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}\ \forall A, B \subseteq \mathcal{X}$. Its conjugate $\overline{P}$ is a maxitive measure.

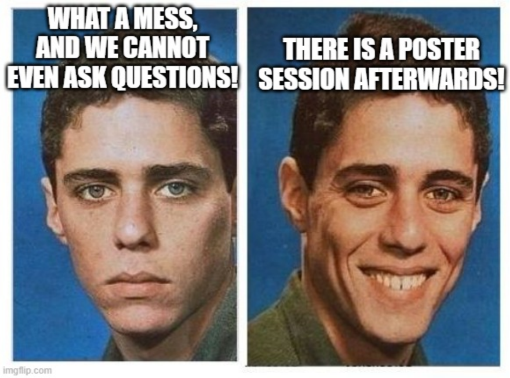
Minitive measures

A lower probability is minitive iff $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}\ \forall A, B \subseteq \mathcal{X}$. Its conjugate $\overline{P}$ is a maxitive measure.

- (Miranda et al. 2015) $\mathcal{A}_C(\overline{P}_1, \overline{P}_2)$ is maxitive iff $\min_i \max_j \overline{P}_i(\{x_j\}) = \max_{j=1,2} \min_{i=1,2} \overline{P}_i(\{x_j\})$ for all $\{x_1, x_2\} \subseteq \mathcal{X}$. ✓
- $\mathcal{A}_D$ is closed in the family of minitive measures. ✓✓
- $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$ is minitive iff $\mathcal{F}_1 \cup \mathcal{F}_2$ is closed under set inclusion. ✓
- $\mathcal{A}_P$ is not closed in the family of minitive measures. ✗

Summary and conclusions

	Conj. $\mathcal{A}_C$	Disj. $\mathcal{A}_D$	Mixt. $\mathcal{A}_M$	Pareto $\mathcal{A}_P$
Comparative	✓✓	✓	✓	✓✓
2-monotone	✓	✓	✓✓	✗
Prob. intervals	✓✓	✓	✓	✓
Belief functions	✗	✓	✓✓	✗
p-boxes	✓✓	✓	✓	✗
Minitive measures	✓	✓✓	✓	✗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