Aggregation of the distortion models induced by the KL divergence and Euclidean distance

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Overview

Overview

Imprecise probability models

Distortion models

Aggregation of Euclidean models

Aggregation of Kullback-Leibler models

Conclusions

Coherent lower prevision

Lower prevision

Gambles: bounded real-valued functions f
Notation: set of gambles $\mathcal{L}(\mathcal{X})$
Lower prevision:

$$\begin{aligned} \underline{P}: \quad \mathcal{L}(\mathcal{X}) &\longrightarrow \mathbb{R} \\ f &\mapsto \underline{P}(f) \end{aligned}$$

Credal set

Credal set: a closed and convex set given by

$$\mathcal{M}(\underline{P}) = \{P \mid E_P(f) \geq \underline{P}(f) \ \forall f\}$$

Coherent lower previsions

Coherence: $\underline{P}$ is coherent if

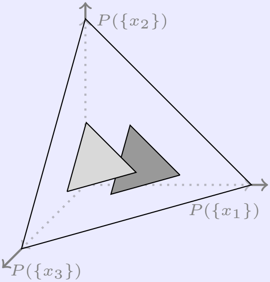
$$\underline{P}(f) = \min\{E_P(f) \mid P \in \mathcal{M}(\underline{P})\} \qquad \forall f \in \mathcal{L}(\mathcal{X})$$

Coherent lower previsions VS credal sets

$$\begin{tikzcd}[column sep=huge] \boxed{\begin{array}{c}\text{Coherent lower}\\ \text{prevision } \underline{P}\end{array}} \arrow[r, bend right, "\mathcal{M} = \{P \mid E_P(f) \geq \underline{P}(f)\ \forall f\}"'] & \boxed{\begin{array}{c}\text{Credal}\\ \text{set } \mathcal{M}\end{array}} \arrow[l, bend right, "\underline{P}(f) = \min\{E_P(f) \mid P \in \mathcal{M}\}"'] \end{tikzcd}$$

Aggregation of coherent lower previsions

Conjunction



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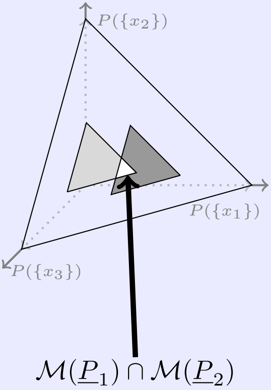

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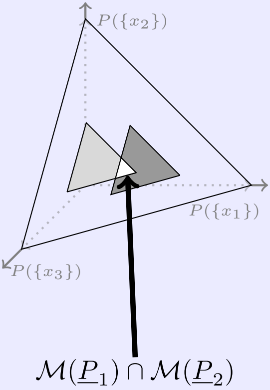
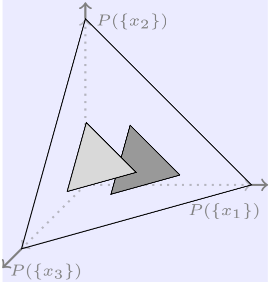
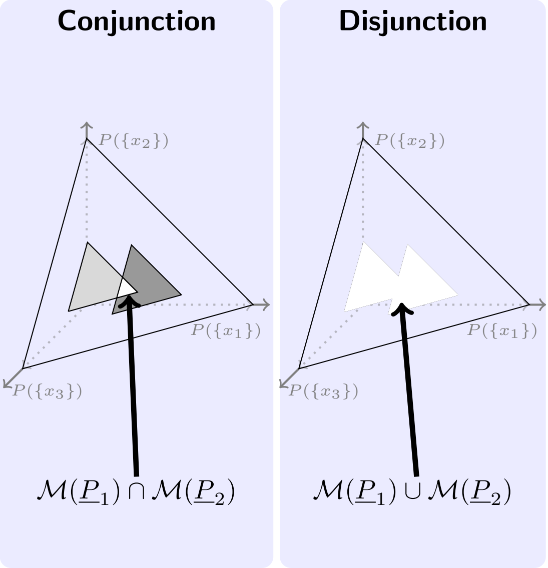


Disjunction



Aggregation of coherent lower previsions



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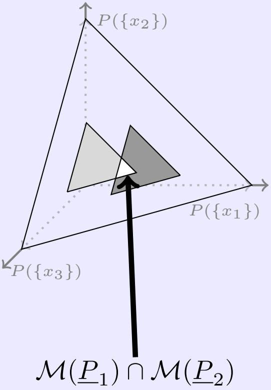
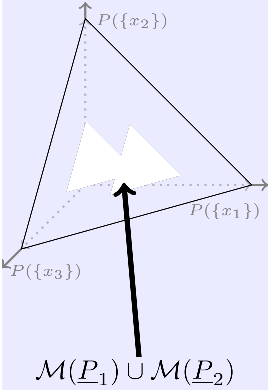
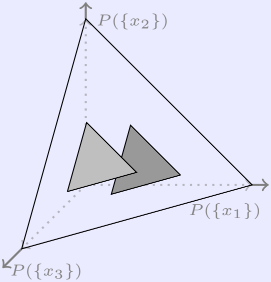


Disjunction

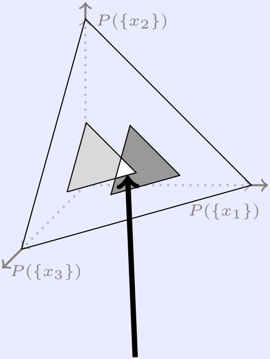


Mixture



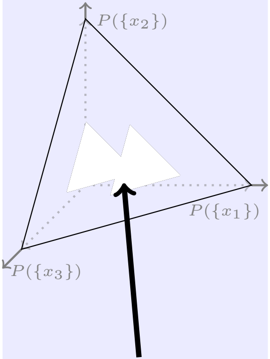
Aggregation of coherent lower previsions

Conjunction



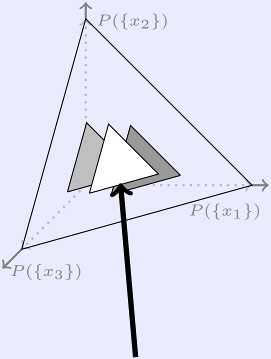
$$\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$$

Disjunction



$$\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$$

Mixture



$$\varepsilon\mathcal{M}(\underline{P}_1) + (1-\varepsilon)\mathcal{M}(\underline{P}_2)$$

$(\varepsilon \in (0,1))$

Overview

Distortion models (Montes et al., 2020)

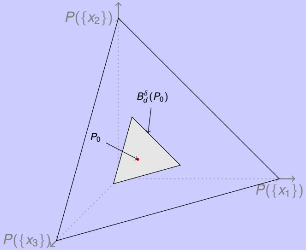
Ingredients

P_0: probability measure

δ: distortion parameter

d: distorting function

Example



The model

$$B_d^\delta(P_0) = \{P \mid d(P, P_0) \leq \delta\}$$

$$\underline{P}_d(f) = \inf\left\{E_P(f) \mid P \in B_d^\delta(P_0)\right\}$$

$$\mathcal{M}(\underline{P}_d) = \{P \mid E_P(f) \geq \underline{P}_d(f)\ \forall f\}$$

d continuous and quasi-convex

Assumptions

$\mathcal{X}$ finite

δ "small"

Particular models

Pari Mutuel Model

$$\underline{P}_{PMM}(A) = \max\{0, (1+\delta)P_0(A) - \delta\}$$

Pelessoni et al., 2010
Montes et al., 2019
Walley, 1991

Linear vacuous model

$$\underline{P}_{LV}(A) = (1-\delta)P_0(A) \qquad A \neq \mathcal{X}$$

Huber, 1981
Walley, 1991

Constant odds ratio

$$\underline{P}_{COR}(A) = \frac{(1-\delta)P_0(A)}{1-\delta P_0(A)}$$

Benavoli and Zaffalon, 2013
Walley, 1991

Total Variation model

$$\underline{P}_{TV}(A) = \max\{0, P_0(A) - \delta\} \qquad A \neq \mathcal{X}$$

Montes et al., 2020
Herron et al., 1997

Vertical barrier models

$$\underline{P}_{VB}(A) = \max\{bP_0(A) + a, 0\} \qquad \begin{array}{l} a \leq 0, b > 0 \\ a + b \in (0,1) \end{array}$$

Pelessoni et al., 2021
Corsato et al., 2019

Increasing transformation

$$\underline{P}(A) = g(P_0(A))$$

$g : [0,1] \to [0,1]$ increasing
$g(0) = 0, g(1) = 1$

Bronevich, 2005

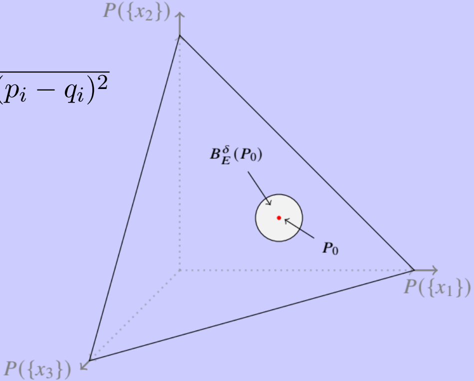
Euclidean model

Euclidean model

$$d_E(P,Q) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$$

$P_0 \longrightarrow B_E^{\delta}(P_0)$

$\delta \longrightarrow B_E^{\delta}(P_0)$



Coherent lower prevision

$f \equiv (a_1, \ldots, a_n)$ $\qquad$ $\underline{P}_E(f) = E_{P_0}(f) - \delta\sqrt{n}S_f$

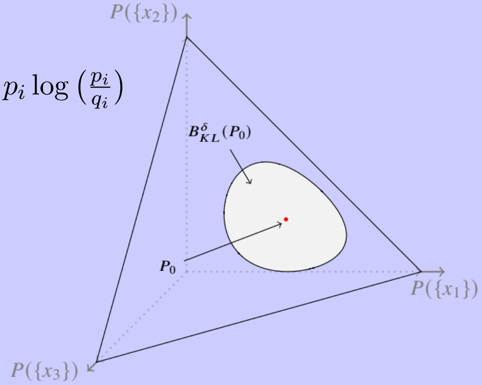
Neighbourhood models induced by the Euclidean distance and the KL divergence. Montes. Submitted.

KL-model

KL-model

$$D_{KL}(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$$

$$\begin{matrix} P_0 \searrow & \\ & B^{\delta}_{KL}(P_0) \\ \delta \nearrow & \end{matrix}$$



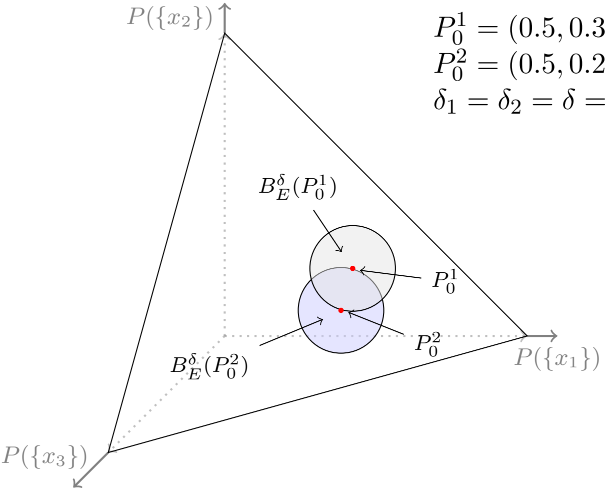
Coherent lower prevision

$$\underline{P}_{KL}(f) = \min\{E_P(f) \mid D_{KL}(P,P_0) = \delta\}$$

Neighbourhood models induced by the Euclidean distance and the KL divergence. Montes. Submitted.

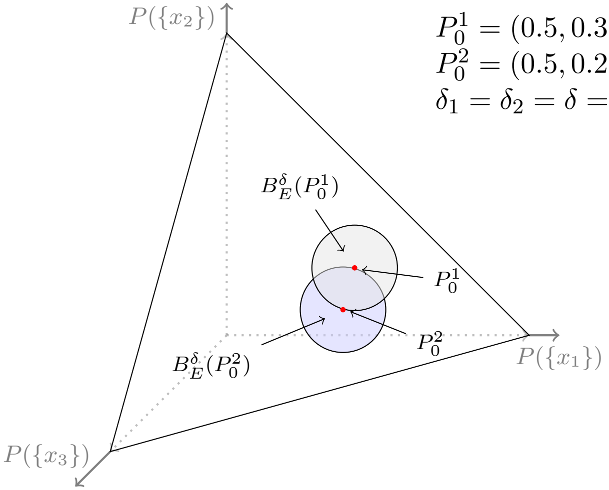
Overview

Conjunction of E-models



$$P_0^1 = (0.5, 0.3, 0.2)$$
$$P_0^2 = (0.5, 0.2, 0.3)$$
$$\delta_1 = \delta_2 = \delta = \sqrt{2}/10$$

Conjunction of E-models



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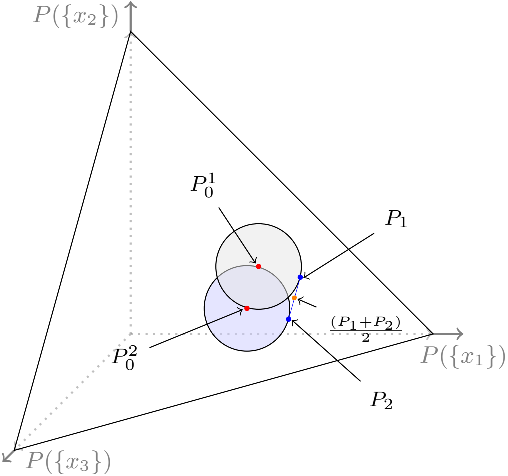
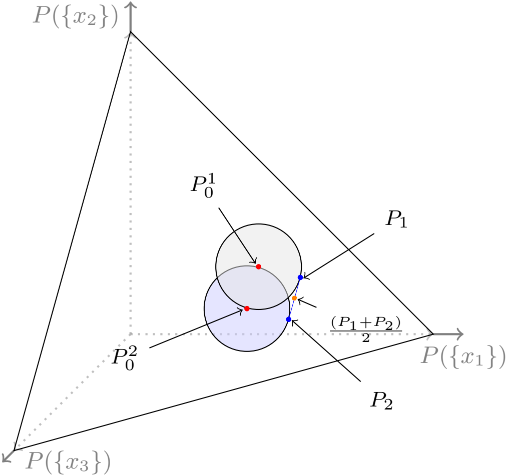
Conclusion: the E-model is not preserved under conjunction

Disjunction of E-models

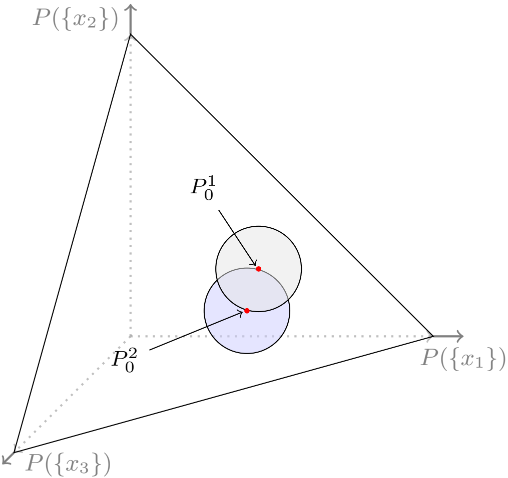


Disjunction of E-models



Conclusion: the E-model is not preserved under disjunction

Disjunction of E-models: unique undominated OA?



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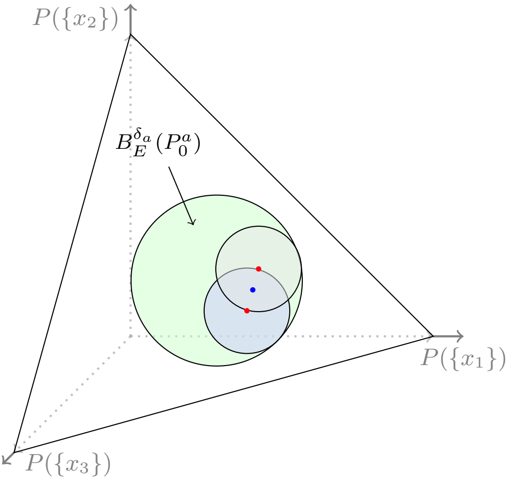

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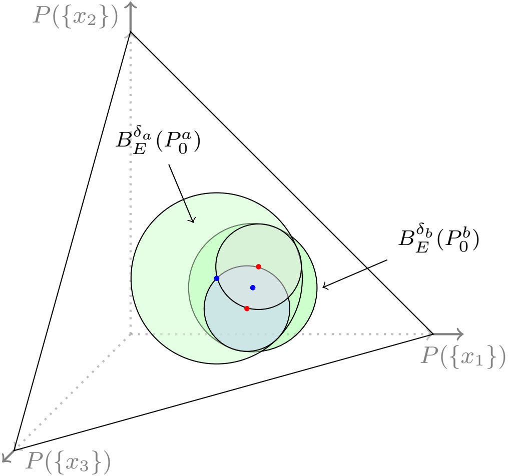
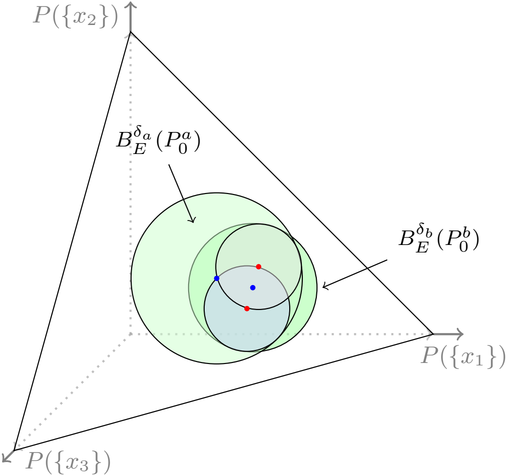


Disjunction of E-models: unique undominated OA?



Conclusion: there is not a unique undominated OA

Mixture of E-models

Proposition: $\varepsilon B_E^{\delta_1}(P_0^1) + (1-\varepsilon) B_E^{\delta_2}(P_0^2) = B_E^{\delta_\varepsilon}(P_0^\varepsilon)$, where:

$$\delta_\varepsilon = \varepsilon\delta_1 + (1-\varepsilon)\delta_2, \qquad P_0^\varepsilon = \varepsilon P_0^1 + (1-\varepsilon) P_0^2.$$

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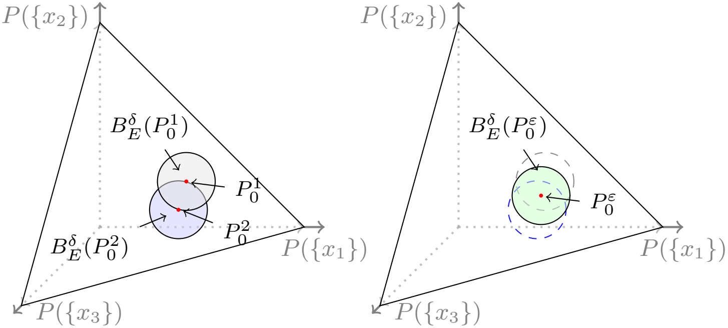
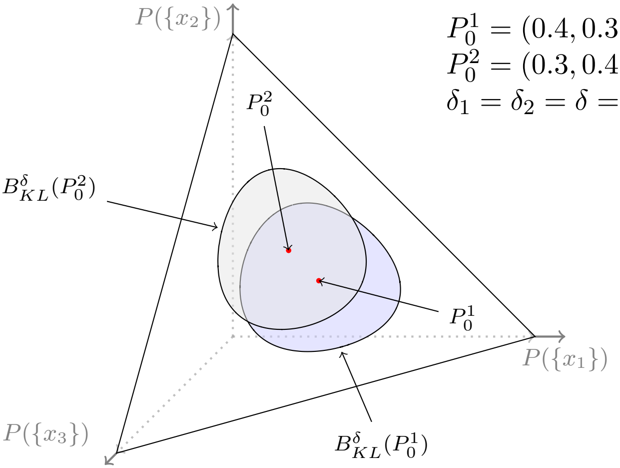


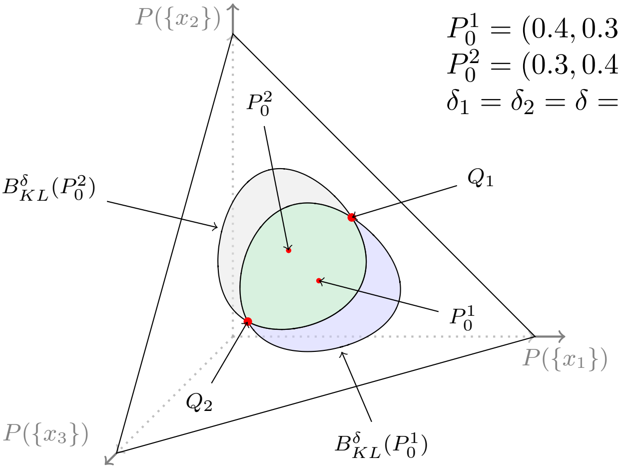
Overview

Conjunction of KL-models



$$P_0^1 = (0.4, 0.3, 0.3)$$
$$P_0^2 = (0.3, 0.4, 0.3)$$
$$\delta_1 = \delta_2 = \delta = 0.1$$

Conjunction of KL-models



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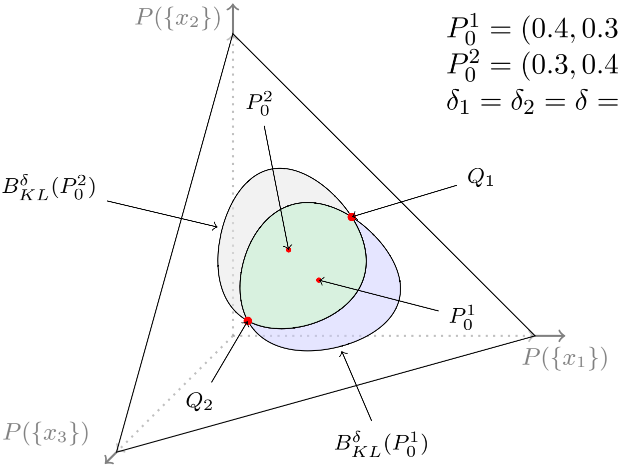
Conjunction of KL-models



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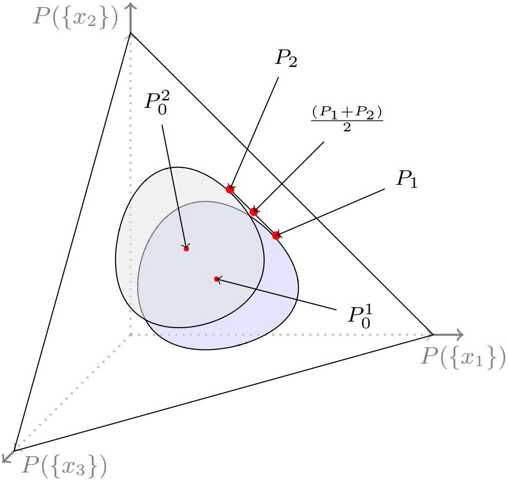
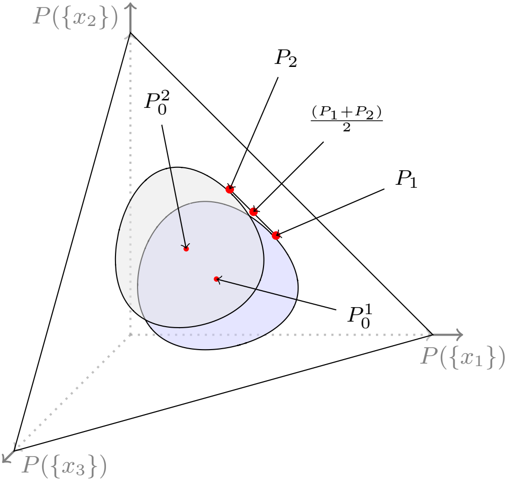
Conclusion: the KL-model is not preserved under conjunction

Disjunction of KL-models

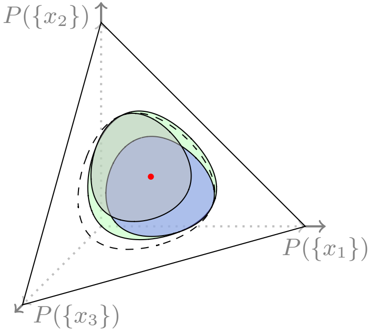


Disjunction of KL-models



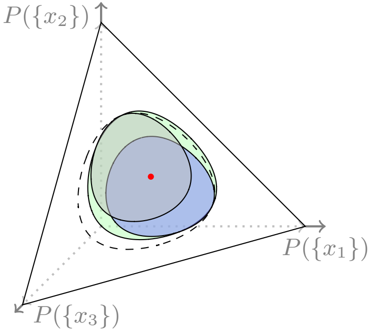
Conclusion: the KL-model is not preserved under disjunction

Disjunction of KL-models: unique undominated OA?



$$P_0 = (0.3552, 0.3552, 0.2896)$$
$$\delta = 0.15381$$

Disjunction of KL-models: unique undominated OA?



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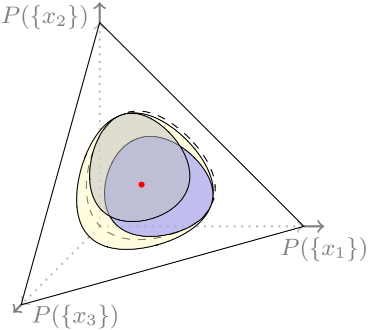
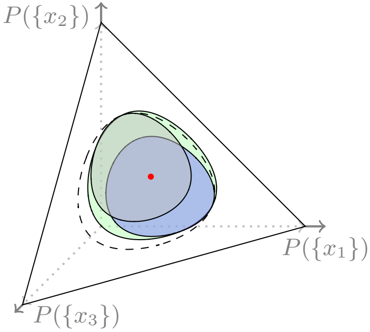


$$P_0 = (1/3, 1/3, 1/3)$$
$$\delta = 0.173594$$

Disjunction of KL-models: unique undominated OA?



$P_0 = (0.3552, 0.3552, 0.2896)$
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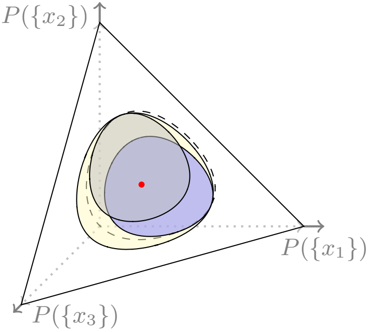
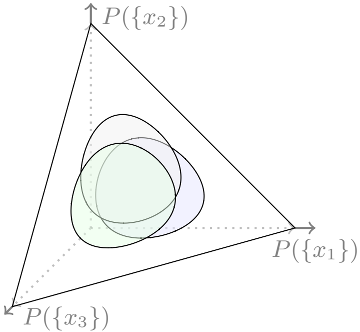


$P_0 = (1/3, 1/3, 1/3)$
$\delta = 0.173594$

Conclusion: there is not a unique undominated OA

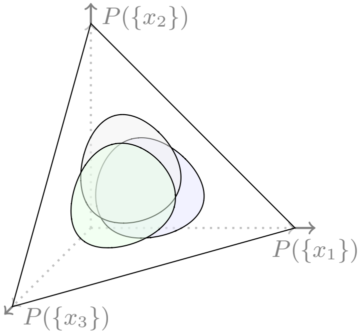
Mixture of KL-models

$$P_0^1 = (0.4, 0.3, 0.3), \;\; P_0^2 = (0.3, 0.4, 0.3), \;\; P_0^3 = (0.3, 0.3, 0.4), \;\; \delta = 0.1$$



Mixture of KL-models

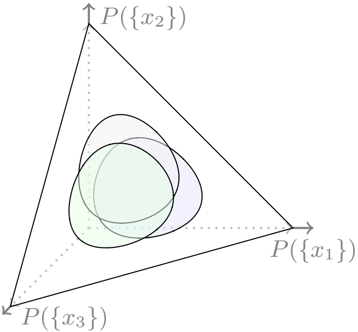
$$P_0^1 = (0.4, 0.3, 0.3), \;\; P_0^2 = (0.3, 0.4, 0.3), \;\; P_0^3 = (0.3, 0.3, 0.4), \;\; \delta = 0.1$$

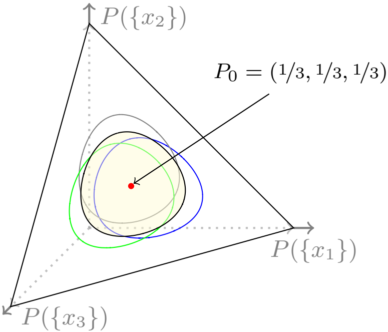


$\underline{P}(\{x_i\}) = 0.112736 \Rightarrow \delta = 0.0986 \Rightarrow \underline{P}(\{x_i, x_j\}) = 0.4499$
$\underline{P}(\{x_i, x_j\}) = 0.44956$

Mixture of KL-models

$$P_0^1 = (0.4, 0.3, 0.3),\;\; P_0^2 = (0.3, 0.4, 0.3),\;\; P_0^3 = (0.3, 0.3, 0.4),\;\; \delta = 0.1$$





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$\underline{P}(\{x_i, x_j\}) = 0.44956$

Conclusion: the KL-model is not preserved under mixtures

Overview

Aggregation: summary

Model	Conjunction	Disjunction	Unique OA	Mixture
PMM	✓	✗	✓	✓
LV	✓	✗	✓	✓
COR	✗	✗	✗	✗
TV	✗	✗	✗	✓
VBM	✗	✗	✗	✓
Euclidean	✗	✗	✗	✓
KL	✗	✗	✗	✗

Processing distortion models: a comparative study. Destercke, Montes, Miranda. IJAR 2022.

Evaluating uncertainty with Vertical Barrier Models. Miranda, Vicig, Pelessoni. IJAR 2024.

Further research

What about combination?

- Marginalisation?
- Joint model?

More about distortion models?

- Wasserstein distance?
- Levy distance?

Applications?

- Statistical inference?
- Credal networks?

Funding

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