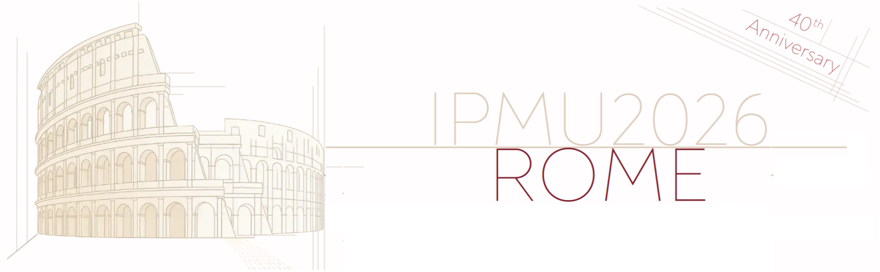


On the commutativity of aggregation and natural extension for lower probabilities

Ignacio Montes **Enrique Miranda**
University of Oviedo

IMAGINE

Aggregation of probabilities

$$\left.\begin{array}{l} \text{Expert 1} \;\to\; P_1 \\ \qquad\quad \vdots \\ \text{Expert } n \;\to\; P_n \end{array}\right\} \longrightarrow \begin{array}{c} \text{Aggregated model} \\ \mathcal{A}(P_1, \ldots, P_n) \end{array}$$

Aggregation of probabilities

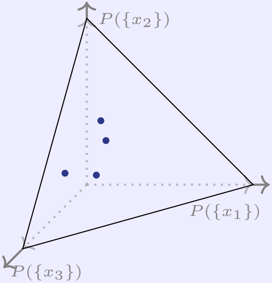
Expert 1 $\rightarrow$ P_1

$\vdots$

Expert n $\rightarrow$ P_n

$\Bigg\} \longrightarrow$ Aggregated model $\mathcal{A}(P_1, \ldots, P_n)$

Mixture



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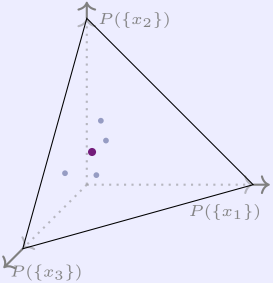

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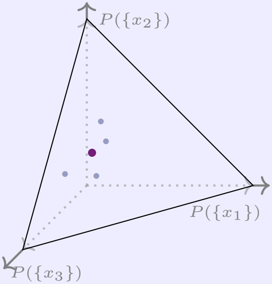
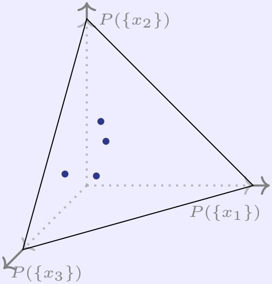


Disjunction



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Aggregated model

$\mathcal{A}(P_1, \ldots, P_n)$

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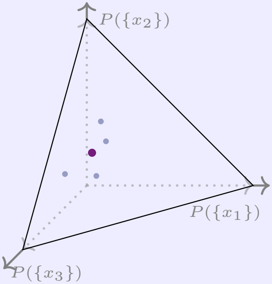

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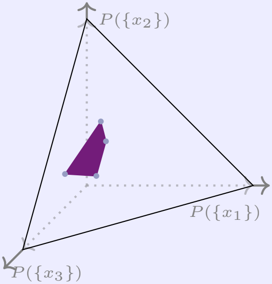

The elicitation and aggregation of beliefs. Walley, 1982.

Aggregation of probabilities

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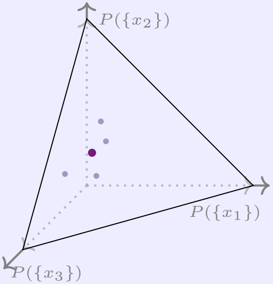

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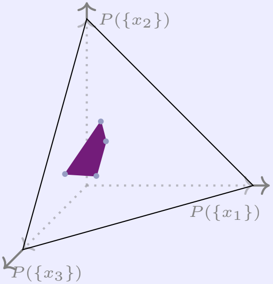

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Overview

Lower probabilities

Lower probability/Capacity

Lower probabilities: A function $\mu : \mathcal{P}(\mathcal{X}) \longrightarrow [0, 1]$ satisfying
1) Monotonicity $(A \subseteq B \Rightarrow \mu(A) \leq \mu(B))$ 2) Normalisation $(\mu(\mathcal{X}) = 1, \mu(\emptyset) = 0)$

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Credal set/Core

Credal set: $\mathcal{M}(\mu) = \{P \mid P(A) \geq \mu(A) \;\; \forall A \subseteq \mathcal{X}\}$

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Coherent lower probability/Exact capacity

Coherence: μ is coherent if

$$\mu(A) = \min\{P(A) \mid P \in \mathcal{M}(\mu)\} \qquad \forall A \subseteq \mathcal{X}$$

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Natural extension

Natural extension: for a gamble $f : \mathcal{X} \to \mathbb{R}$,

$$\underline{E}_{\mu}(f) = \min\{E_P(f) \mid P \in \mathcal{M}(\mu)\}$$

Properties of lower probabilities

2-monotone/Supermodular

μ is 2-monotone if $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$

$\underline{E}$ is 2-monotone if $\underline{E}(f \vee g) + \underline{E}(f \wedge g) \geq \underline{E}(f) + \underline{E}(g)$

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Complete monotonicity/Belief function

μ is complete monotone if for any $p \geq 2$ and $A_1, \ldots, A_p$:

$$\mu\Big(\bigcup_{i=1}^{p} A_i\Big) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (-1)^{|I|+1} \mu\Big(\bigcap_{i \in I} A_i\Big)$$

$\underline{E}$ is complete monotone if for any $p \geq 2$ and $f_1, \ldots, f_p$:

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Description of the problem (I)

$\boxed{\text{$\mu_1, \ldots, \mu_n$ coherent lower probabilities}}$

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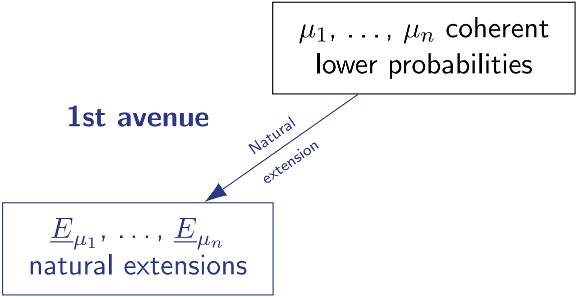

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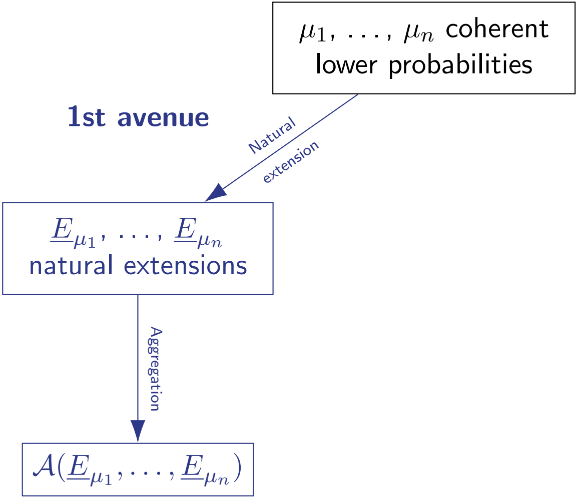

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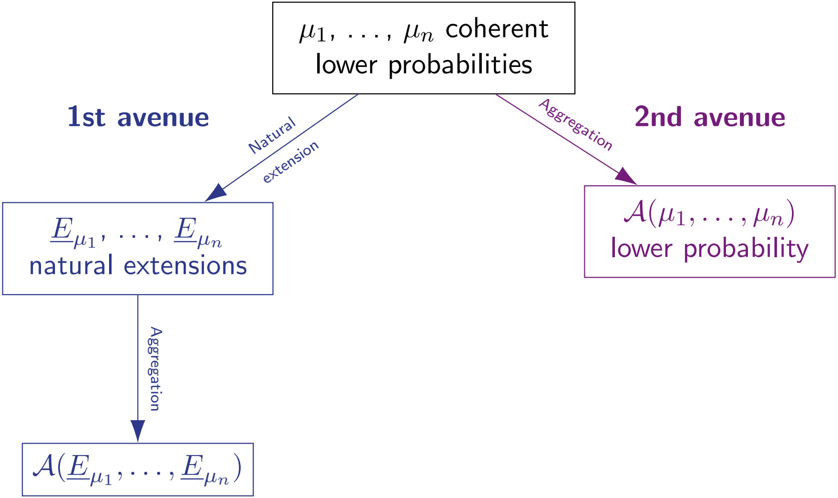

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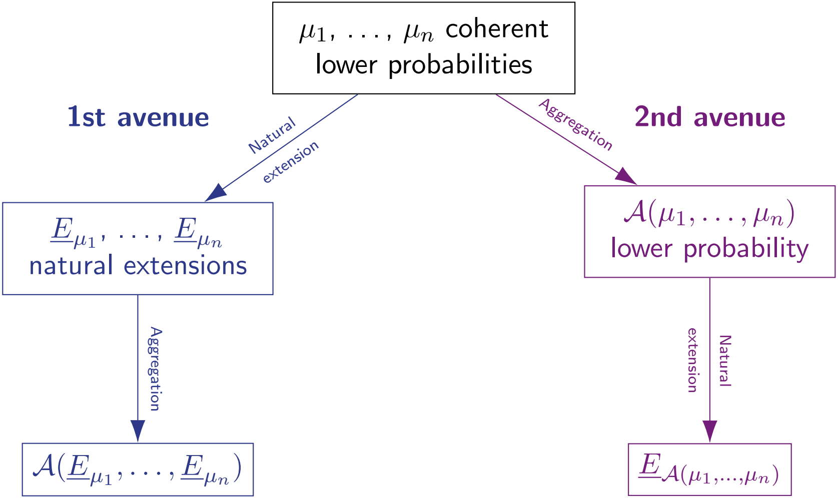

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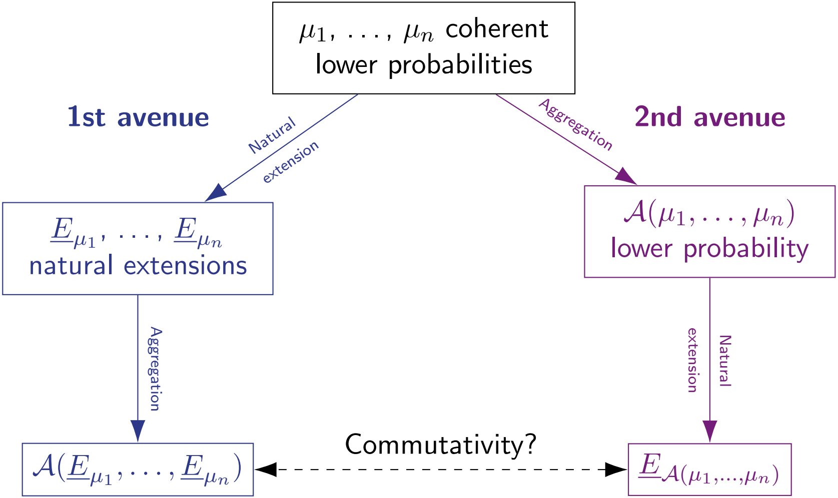


Description of the problem (II)

Preservation of properties

2-monotonicity preservation: If $\mu_1, \ldots, \mu_n$ are 2-monotone, are $\mathcal{A}(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n})$ or $\underline{E}_{\mathcal{A}(\mu_1,\ldots,\mu_n)}$ 2-monotone?

Complete monotonicity preservation: If $\mu_1, \ldots, \mu_n$ are completely monotone, are $\mathcal{A}(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n})$ or $\underline{E}_{\mathcal{A}(\mu_1,\ldots,\mu_n)}$ complete monotone?

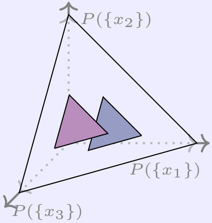
Overview

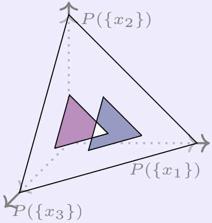
Conjunction rule

Conjunction

$$\mathcal{A}_C(\underline{E}_1, \ldots, \underline{E}_n)(f) = \min\left\{E_P(f) \mid E_P \geq \max\{\underline{E}_1, \ldots, \underline{E}_n\}\right\}$$

Graphical interpretation





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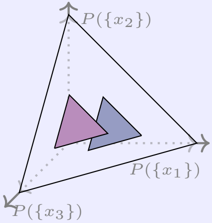


Conjunction of lower probabilities

$$\mathcal{A}_C(\mu_1, \ldots, \mu_n)(A) = \min \left\{ P(A) \mid P \geq \max\{\mu_1, \ldots, \mu_n\} \right\}$$

Conjunction rule: results

Commutativity: $\mathcal{A}_C(\underline{E}_{\mu_1}, \dots, \underline{E}_{\mu_n}) = \underline{E}_{\mathcal{A}_C(\mu_1,\dots,\mu_n)}$ ✓✓

Preservation:

1) 2- and complete monotonicity are not preserved ✗

2) If $\mathcal{M}(\mu_1) \cup \mathcal{M}(\mu_2)$ is convex, 2-monotonicity is preserved ✓

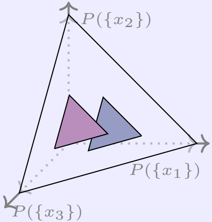
3) The same property does not hold for $n > 2$ or for complete monotonicity ✗

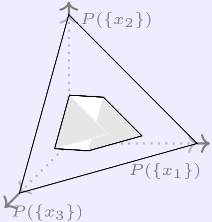
Disjunction rule

Disjunction

$$\mathcal{A}_D(\underline{E}_1, \ldots, \underline{E}_n)(f) = \min\{\underline{E}_1(f), \ldots, \underline{E}_n(f)\}$$

Graphical interpretation





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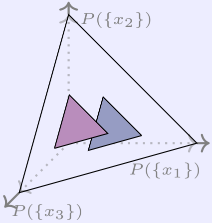

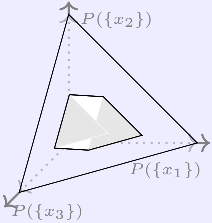

Disjunction of lower probabilities

$$\mathcal{A}_D(\mu_1, \ldots, \mu_n)(A) = \min\{\mu_1(A), \ldots, \mu_n(A)\}$$

Disjunction rule: results

Commutativity: $\mathcal{A}_D(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}) \geq \underline{E}_{\mathcal{A}_D(\mu_1,\ldots,\mu_n)}$ ✓

Preservation: 2- and complete monotonicity are not preserved ✗

Disjunction rule: results

Commutativity: $\mathcal{A}_D(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}) \geq \underline{E}_{\mathcal{A}_D(\mu_1,\ldots,\mu_n)}$ ✓

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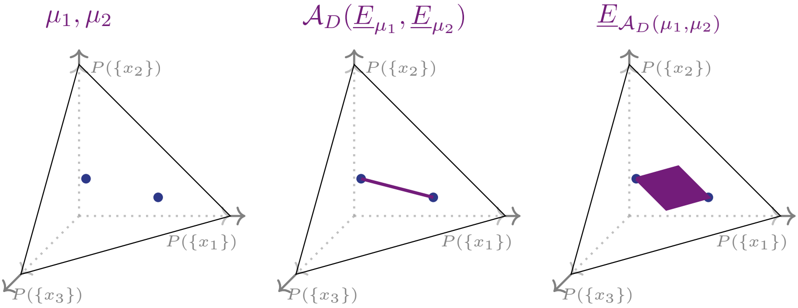
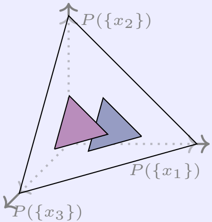


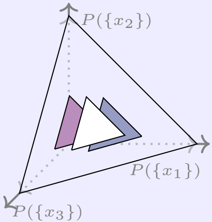
Mixture rule

Mixture

$$\mathcal{A}_M(\underline{E}_1, \ldots, \underline{E}_n)(f) = \alpha_1 \underline{E}_1(f) + \ldots + \alpha_n \underline{E}_n(f)$$

Graphical interpretation





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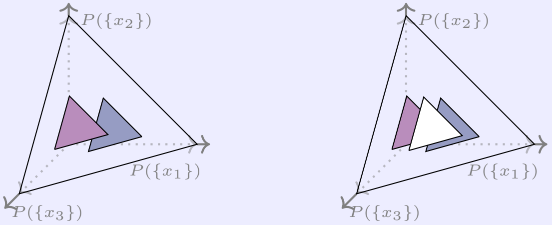


Mixture of lower probabilities

$$\mathcal{A}_M(\mu_1, \ldots, \mu_n)(A) = \alpha_1 \mu_1(A) + \ldots + \alpha_n \mu_n(A)$$

Mixture rule: results

Commutativity: $\mathcal{A}_M(\underline{E}_{\mu_1}, \dots, \underline{E}_{\mu_n}) \geq \underline{E}_{\mathcal{A}_M(\mu_1,\dots,\mu_n)}$ ✓

Preservation:

1) 2- and complete monotonicity are preserved. . . ✓✓

2) . . . and if $\mu_1, \dots, \mu_n$ are 2-monotone, then commutativity holds: $\mathcal{A}_M(\underline{E}_{\mu_1}, \dots, \underline{E}_{\mu_n}) = \underline{E}_{\mathcal{A}_M(\mu_1,\dots,\mu_n)}$ ✓✓

Overview

Summary

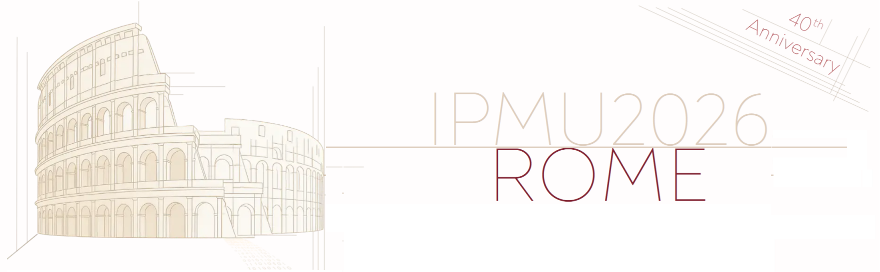
Commutativity: $\mathcal{A}\left(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}\right) = \underline{E}_{\mathcal{A}(\mu_1,\ldots,\mu_n)}$?

Property preservation: Do these two avenues preserve the properties of $\mu_1, \ldots, \mu_n$?

Rule	Commutativity?	2-monotonicity preservation?	complete-monotonicity preservation?
$\mathcal{A}_C$	$\mathcal{A}_C\left(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}\right) = \underline{E}_{\mathcal{A}_C(\mu_1,\ldots,\mu_n)}$	Yes *(if $n = 2$ and $\mathcal{M}(\mu_1) \cup \mathcal{M}(\mu_2)$ is convex)*	No
$\mathcal{A}_D$	$\mathcal{A}_D\left(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}\right) \geq \underline{E}_{\mathcal{A}_D(\mu_1,\ldots,\mu_n)}$	No	No
$\mathcal{A}_M$	$\mathcal{A}_M\left(\underline{E}_{\mu_1}, \ldots, \underline{E}_{\mu_n}\right) \geq \underline{E}_{\mathcal{A}_M(\mu_1,\ldots,\mu_n)}$	Yes *(and both approaches coincide)*	Yes *(and both approaches coincide)*

Funding

This contribution is part of grant PID2022-140585NB-I00 funded by MICIU/AEI/10.13039/501100011033 and “FEDER/UE”.



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