Decision Making with Generalised Distortion Models

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Imprecision in the Aggregation of information provided by experts (IMAGINE)

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Index

Index

Motivation

Decision Theory → Mathematically formulate **rational choice criteria** → **Make risk-assessed decisions**

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Traditionally rely on precise probabilistic info.

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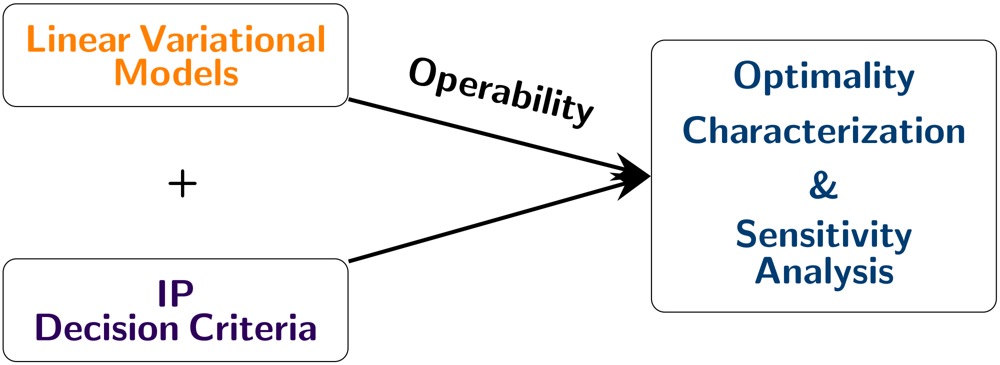
Unfit to model partial beliefs or context indeterminacies → **IPs** → **Make informed decisions robustified against uncertainty**



Troffaes (2007)

Imprecise Probabilities: Different models permitting robust statistical analysis based on sets of probability measures, which we call **credal sets** (*Imprecise beliefs*) Walley (1991).

Consider a finite probability space Ω and denote with $\mathbb{P}(\Omega)$ the set of prob. measures and

$$\mathcal{L}(\Omega) = \{f : \Omega \to \mathbb{R}\} \quad - \quad \textbf{Set of Gambles}$$

Lower Prevision

A *lower prevision* assigns a **lower bound on the expectation operator** compatible with our beliefs

$$\underline{P} : \mathcal{L}(\Omega) \longrightarrow \mathbb{R}$$

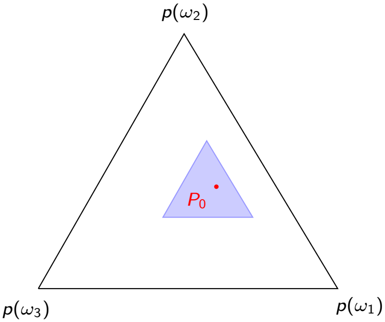
The **compatible probability measures**, $\mathcal{M}(\underline{P})$, satisfy

$$P \in \mathcal{M}(\underline{P}) \Leftrightarrow \underline{P}(f) \leq P(f) \quad \forall f \in \mathcal{L}(\Omega)$$

Coherent Lower Previsions

We say that $\underline{P}$ **is coherent** if the lower bounds $\underline{P}(f)$ are attainable:

$$\underline{P}(f) = \min_{P \in \mathcal{M}(\underline{P})} P(f) \quad \forall f \in \mathcal{L}(\Omega)$$



Distortion Models

Given $P_0 \in \mathbb{P}(\Omega)$, $\delta \geq 0$ and a distortion function $d : \mathbb{P}(\Omega) \times \mathbb{P}(\Omega) \to \mathbb{R}^+$, the *distortion model* associated with (P_0, d, δ) is

$$B_d^\delta(P_0) = \{P \in \mathbb{P}(\Omega) \,|\, d(P, P_0) \leq \delta\}.$$

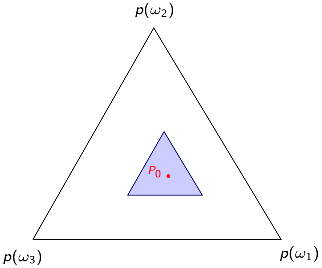
Popular Models i

Consider $P_0 \in \mathbb{P}(\Omega)$ and $\delta \geq 0$, for any $A \neq \Omega, \emptyset$,

Linear Vacuous Models

$$\underline{P}_{LV}(A) = (1-\delta)\, P_0(A).$$

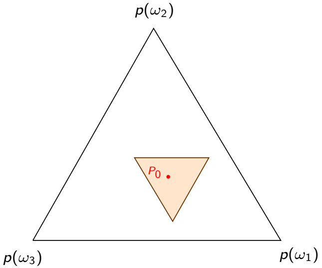
'There is proportion δ of contaminated data coming from an unknown source'



Pari Mutuel Models

$$\underline{P}_{PM}(A) = \max\{(1+\delta)\, P_0(A) - \delta, 1\}.$$

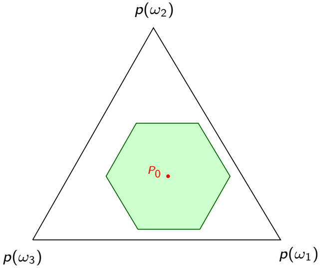
'There is a relative taxation to bets on events given by δ'



Total Variation Models

$$\underline{P}_{TV}(A) = \max\{P_0(A) - \delta, 0\}.$$

'There is a maximal difference in the estimated probabilities across events given by δ'



Popular Models ii

All the presented models can equivalently be expressed via lower previsions.

It was proven that, if $P_0 \in \mathbb{P}^*(\Omega)$, their associated lower previsions are given by:

$$\underline{P}_{LV}(f) = P_0(f) - \delta\left(P_0(f) - \min f\right) \quad \text{if } \delta < 1.$$

$$\underline{P}_{PM}(f) = P_0(f) - \delta\left(\max f - P_0(f)\right) \quad \text{if } \delta < \min_{\omega \in \Omega} \frac{P_0(\{\omega\})}{1 - P_0(\{\omega\})}.$$

$$\underline{P}_{TV}(f) = P_0(f) - \delta\left(\max f - \min f\right) \quad \text{if } \delta < \min_{\omega \in \Omega} P_0(\{\omega\}).$$

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$$\left.\begin{aligned}
\underline{P}_{LV}(f) &= P_0(f) - \delta\,(P_0(f) - \min f) && \text{if } \delta < 1.\\
\underline{P}_{PM}(f) &= P_0(f) - \delta\,(\max f - P_0(f)) && \text{if } \delta < \min_{\omega\in\Omega} \tfrac{P_0(\{\omega\})}{1-P_0(\{\omega\})}.\\
\underline{P}_{TV}(f) &= P_0(f) - \delta\,(\max f - \min f) && \text{if } \delta < \min_{\omega\in\Omega} P_0(\{\omega\}).
\end{aligned}\right\}$$

Reduced precise assessments via a δ-scaled variability related term

LVar Models i

Consider $P_0 \in \mathbb{P}^*(\Omega)$, $\delta \geq 0$ and a *Variational Operator* $\underline{\Delta} : \mathcal{L}(\Omega) \to [0, +\infty)$ satisfying

> **1.** $\underline{\Delta} f = 0 \Leftrightarrow f$ is a constant gamble.
>
> **2.** $\delta \underline{\Delta} f \leq P_0(f) - \min f$ for any $f \in \mathcal{L}(\Omega)$.
>
> **3.** $\underline{\Delta}(\alpha f) = \alpha \underline{\Delta} f$ for any $\alpha \geq 0$.
>
> **4.** $\underline{\Delta}(f + g) \leq \underline{\Delta} f + \underline{\Delta} g$ for any $f, g \in \mathcal{L}(\Omega)$.

Linear Variational Models

We call *Linear Variational* (LVar) model associated with $(P_0, \underline{\Delta}, \delta)$ to the lower prevision given by:

$$\underline{P}_{\underline{\Delta}}(f) = P_0(f) - \delta \, \underline{\Delta} f, \quad \forall f \in \mathcal{L}(\Omega).$$

An LVar model $\underline{P}_{\Delta}$ is **coherent** and satisfies $P_0 \in \mathcal{M}(\underline{P}_{\Delta})$.

Condition 2 establishes a **bound on** δ such that the LVar model is coherent:

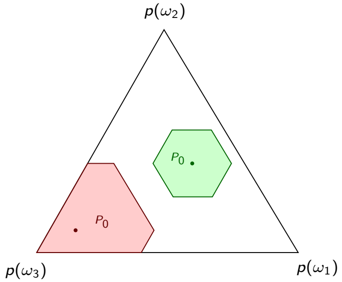
$$\delta \leq \inf_{f \in \mathcal{L}^*(\Omega)} \frac{P_0(f) - \min f}{\underline{\Delta} f}.$$

The LV, PM and TV are LVar models:

$$\underline{\Delta}_{LV}(f) = P_0(f) - \min f$$

$$\underline{\Delta}_{PM}(f) = \max f - P_0(f)$$

$$\underline{\Delta}_{TV}(f) = \max f - \min f$$



Element	
$\Omega = \{\omega_i\}_{i=1}^n$	Scenarios
$D = \{d_j\}_{j=1}^m$	Alternatives
$\{J_{d_j}\}_{j=1}^m \subseteq \mathcal{L}(\Omega)$	Gambles representing the decisions' utility

Decision	ω_1 (Rains)	ω_2 (Doesn't)
d_1 (Umbrella)	0	-1
d_2 (Don't)	-5	0

No Info. $\longrightarrow$ $\text{opt}_{\geq}(D) = \{d \in D \mid J_e \not\geq J_d \; \forall e \in D\} (\equiv D)$

Prob Info. $\xrightarrow{\text{Savage}}$ $\text{opt}_P(D) = \{d \in D \mid P(J_d) \geq P(J_e) \; \forall e \in D\}$

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A middle-ground is necessary!

Let $\underline{P}$ be a coherent lower prevision, the **Expected Utility** can be extended as in Troffaes (2007):

Γ-maximin $\longrightarrow$ $\text{opt}_{\underline{P}} = \{d \in D \mid \underline{P}(J_d) \geq \underline{P}(J_e)\ \forall e \in D\}$

Γ-maximax $\longrightarrow$ $\text{opt}_{\overline{P}} = \left\{d \in D \mid \overline{P}(J_d) \geq \overline{P}(J_e)\ \forall e \in D\right\}$

E-admissibility $\longrightarrow$ $\text{opt}_{\mathcal{M}(\underline{P})} = \{d \in D \mid \exists P \in \mathcal{M}(\underline{P}) \text{ s.t. } d \in \text{opt}_P\}$

Maximality $\longrightarrow$ $\text{opt}_{\geq_{\underline{P}}} = \{d \in D \mid \underline{P}(J_e - J_d) \leq 0\ \forall e \in D\}$

Interval Dominance $\longrightarrow$ $\text{opt}_{\sqsupset_{\underline{P}}} = \left\{d \in D \mid \overline{P}(J_d) \geq \underline{P}(J_e)\ \forall e \in D\right\}$

Depending on the criterion employed, the dominance relationships between alternatives might be easier to assess, or rather tend to incomparability.

Relationships between criteria

Therefore, some criteria might be more lenient than other:

Lenient

$$\begin{array}{ccccccc} \mathrm{opt}_{\overline{P}} & \subseteq & \mathrm{opt}_{\mathcal{M}(\underline{P})} & \subseteq & \mathrm{opt}_{\geq_{\underline{P}}} & \subseteq & \mathrm{opt}_{\sqsupseteq_{\underline{P}}} \\ & & & & \rotatebox{90}{$\subseteq$} & & \\ & & & & \mathrm{opt}_{\underline{P}} & & \end{array}$$

Strict

Employing an LVar model $\underline{P}_{\underline{\Delta}}$ leads to the reformulation of each of the presented IP criteria as **discounted precise utility expectations**:

$$d \in \mathrm{opt}_{\underline{P}_{\underline{\Delta}}} \iff P_0(J_d) \geq \max_{e \in D} \Big\{ P_0(J_e) - \delta \big(\underline{\Delta} J_e - \underline{\Delta} J_d \big) \Big\},$$

$$d \in \mathrm{opt}_{\overline{P}_{\underline{\Delta}}} \iff P_0(J_d) \geq \max_{e \in D} \Big\{ P_0(J_e) - \delta \big(\overline{\Delta} J_d - \overline{\Delta} J_e \big) \Big\},$$

$$d \in \mathrm{opt}_{\mathcal{M}(\underline{P}_{\underline{\Delta}})} \iff P_0(J_d) \geq \max_{\lambda \in G(D)} \Big\{ P_0(J_\lambda) - \delta \, \underline{\Delta}(J_\lambda - J_d) \Big\},$$

$$d \in \mathrm{opt}_{\geq_{\underline{P}_{\underline{\Delta}}}} \iff P_0(J_d) \geq \max_{e \in D} \Big\{ P_0(J_e) - \delta \, \underline{\Delta}(J_e - J_d) \Big\},$$

$$d \in \mathrm{opt}_{\sqsupseteq_{\underline{P}_{\underline{\Delta}}}} \iff P_0(J_d) \geq \max_{e \in D} \Big\{ P_0(J_e) - \delta \, \big(\underline{\Delta} J_e + \overline{\Delta} J_d \big) \Big\}.$$

The prior characterizations permit analyzing when an alternative is never optimal or the amount of distortion required so as it is deemed optimal according to certain IP criteria.

Γ-maximin

Given $d \in D$, if there exists $e \in \mathrm{opt}_{\geq}$ such that

$$P_0(J_d) \leq P_0(J_e) \text{ and } \underline{\Delta} J_d \geq \underline{\Delta} J_e,$$

with some of the inequalities being strict, then $d \notin \mathrm{opt}_{\underline{P}_\Delta}$ for any $\delta > 0$.
Else, $d \in \mathrm{opt}_{\underline{P}_\Delta}$ if and only if

$$\max_{e \in D^{d-}_{\underline{P}_\Delta}} \left\{ \frac{P_0(J_e) - P_0(J_d)}{\underline{\Delta} J_e - \underline{\Delta} J_d} \right\} \leq \delta \leq \min_{e \in D^{d+}_{\underline{P}_\Delta}} \left\{ \frac{P_0(J_e) - P_0(J_d)}{\underline{\Delta} J_e - \underline{\Delta} J_d} \right\},$$

where $D^{d-}_{\underline{P}_\Delta} = \{ e \in D \,|\, \underline{\Delta} J_d < \underline{\Delta} J_e \}, \quad D^{d+}_{\underline{P}_\Delta} = \{ e \in D \,|\, \underline{\Delta} J_d > \underline{\Delta} J_e \}.$

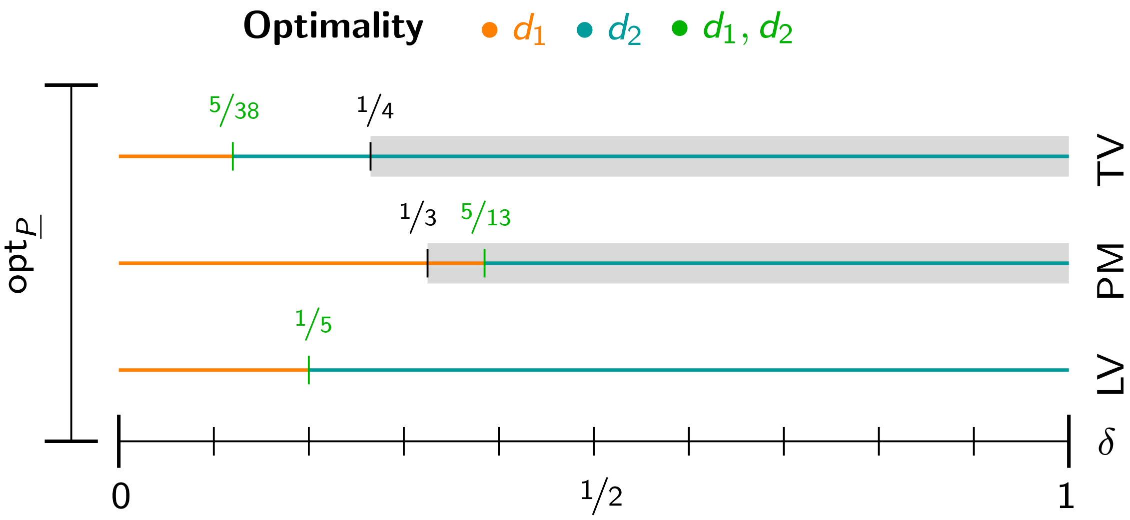
Index

Illustrative example i

$$\Omega = \{\omega_1, \omega_2, \omega_3\}, \quad P_0 = (1/2, 1/4, 1/4)$$

	ω_1	ω_2	ω_3	$P_0(J_{d_i})$	$\underline{\Delta}_{LV}(J_{d_i})$	$\underline{\Delta}_{PM}(J_{d_i})$	$\underline{\Delta}_{TV}(J_{d_i})$
J_{d_1}	3	-3	0	$3/4$	$15/4$	$9/4$	6
J_{d_2}	$3/4$	$-1/2$	$-1/2$	$1/8$	$5/8$	$5/8$	$5/4$

- $\mathrm{opt}_{P_0} = \{d_1\}$
- Since $P_0(J_{d_2}) < P_0(J_{d_1})$ and $\overline{\Delta}(J_{d_2}) < \overline{\Delta}(J_{d_1})$ for all the models, d_2 will never be Γ-maximax $\Longrightarrow$ $\mathrm{opt}_{\overline{P}} = \{d_1\}$ for all models and δ.


Optimality
d_1
d_2
d_1, d_2
$\mathrm{opt}_{\underline{P}}$
5/38
1/4
TV
1/3
5/13
PM
1/5
LV
δ
0
1/2
1

Index

Conclusions

- Introduced **LVar models**, a **new family of IP models** which **generalize prominent frameworks**.
- Doing Decision Theory with LVar models permits **sensitivity analysis** and **discounted expected-utility comparisons**.
- **Further research:**
  - Properties of LVar models
  - Beyond δ's bound
  - Further optimality criteria
  - Other applications

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