An order-theoretic study of the credal sets generated by comparative probabilities

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Summary

- We study the complexity of the credal set determined by a **comparative probability order** on the singletons.
- We characterise the set of **extreme points** and provide tighter bounds on their cardinality.
- We analyse properties of the associated **lower probability**.

What are comparative probabilities?

What will the weather be like tomorrow?

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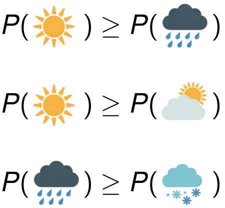
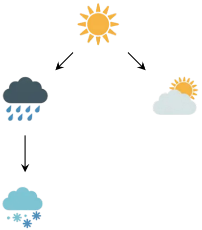
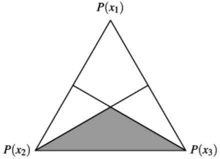




Extreme points of the credal set

- **Credal set:** convex set



- **Extreme points:**

$$P \neq \alpha P_1 + (1-\alpha) P_2 \quad \alpha \in (0,1).$$

The extreme points of the set allow us to:

- Characterise the set using **convex combinations.**
- Determine the **lower and upper probabilities and previsions** that summarise it.
- Update the set by **conditioning.**

How many extreme points are there? What is their behavior?

Problem formulation

- **Finite possibility space:**

$$\mathscr{X} = \{x_1, \ldots, x_n\}$$

- **Binary relation:**

$$x_i \succsim x_j \iff P(\{x_i\}) \geq P(\{x_j\})$$

$$\mathscr{L} = \{(i,j) : x_i \succsim x_j\}$$

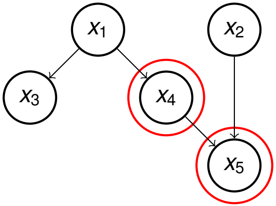
- **Credal set:**

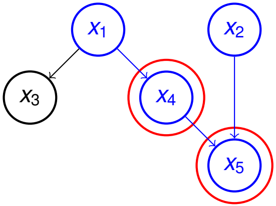
$$\mathscr{P}(\mathscr{L}) = \{P \in \mathbb{P}_{\mathscr{X}} : P(\{x_i\}) \geq P(\{x_j\}), \ \forall (i,j) \in \mathscr{L}\}$$

- **Graph:**

$$\mathscr{G} = (\mathscr{X}, \mathscr{L})$$

Sets closed by ancestors





- **Ancestors of a set:**

$$\langle B \rangle_{\mathscr{L}} := \{x_i \in \mathscr{X} : \exists x_j \in B \text{ such that } x_i \succeq x_j\}$$

- **Sets closed by ancestors:**

$$\mathscr{H}_{\mathscr{L}} := \{\langle B \rangle_{\mathscr{L}} : B \subseteq \mathscr{X}\}$$

Computation of the extreme points

Theorem (Miranda and Destercke, 2015)

Let $\mathscr{L}$ be an acyclic relation, and let $\mathscr{P}(\mathscr{L})$ be the credal set it. For any $B \subseteq \mathscr{X}$, let P_B denote the uniform probability measure on the set B. Then $\text{Ext}(\mathscr{P}(\mathscr{L}))$ is given by

$$\{P_B : B \in \mathscr{H}, \nexists B_1, B_2 \in \mathscr{H} \text{ such that } B_1 \cup B_2 = B \text{ and } B_1 \cap B_2 = \emptyset\}.$$

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$$P_{\{x_1,x_3\}} = (\mathbf{0.5}, 0, \mathbf{0.5}, 0) \iff B = \{x_1, x_3\}$$

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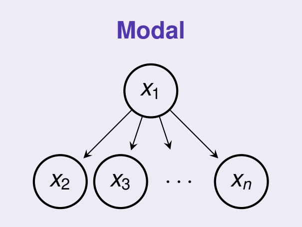
Extreme point of the credal set $\iff$ **Connected set closed by ancestors**

Types of relations

- **Bounded-above relations:** Unique maximal element x_1



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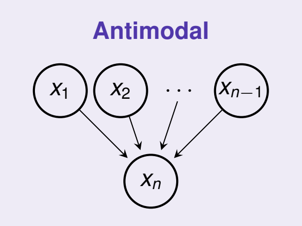

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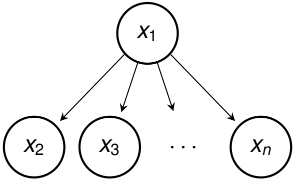
Bound on the number of extreme points

Theorem (Miranda and Destercke, 2015)

The maximal number of extreme points of $\mathscr{P}(\mathscr{L})$ is $2^{|\mathscr{X}|-1}$.



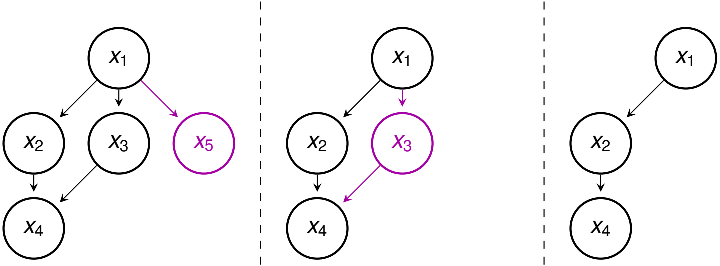
Bounded-above relations

Lemma

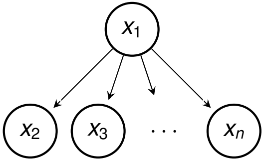
$Ext(\mathscr{P}(\mathscr{L})) = \mathscr{H}$ if and only if $\mathscr{L}$ is bounded above.

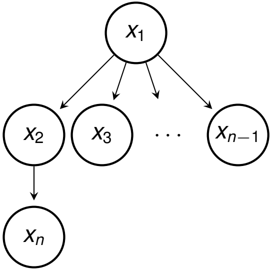
Theorem

The number of extreme points of $\mathscr{P}(\mathscr{L})$ of a bounded-above relation $\mathscr{L}$ is $2^{|\mathscr{X}|-1}$ if and only if the relation is modal. If the relation $\mathscr{L}$ is non modal, the maximum number of extreme points of $\mathscr{P}(\mathscr{L})$ is $3 \cdot 2^{|\mathscr{X}|-3}$.



Bounded-above relations





$$\mathrm{Ext}(\mathscr{P}(\mathscr{L})) = 2^{n-1}$$

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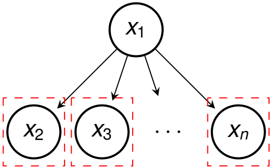
Bounded-above relations



$n-1$

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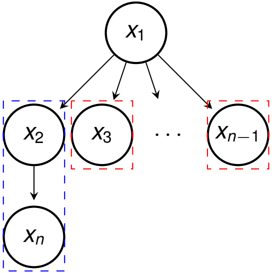
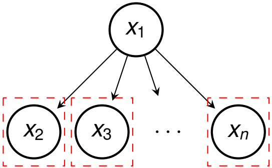
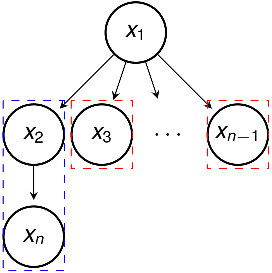


1 $\qquad$ $n-3$

$$\mathrm{Ext}(\mathscr{P}(\mathscr{L})) = 3^1 \cdot 2^{n-3}$$

Bounded-above relations





$n-1$ | 1 | $n-3$

$\text{Ext}(\mathscr{P}(\mathscr{L})) = 2^{n-1}$ | $\text{Ext}(\mathscr{P}(\mathscr{L})) = 3^1 \cdot 2^{n-3}$

$$B \in \mathscr{H}_{\mathscr{L}} \iff x_1 \in B \quad \text{and} \quad B \cap \mathscr{X}_i \in \mathscr{H}_{\mathscr{L}_i} \cup \{\emptyset\}$$

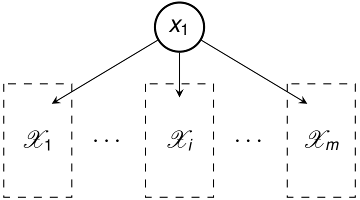
Bounded-above relations

Proposition

Let $\mathscr{L}$ be a bounded-above relation with maximal element x_1, and let $\{\mathscr{X}_i\}_{i=1}^m$ be the connected components of $\mathscr{L}_{x_1}$. Denote by $\mathscr{L}_i$ the restriction of $\mathscr{L}$ to $\mathscr{X}_i$. Then the following mapping f

$$\begin{aligned} f : \mathscr{H}_{\mathscr{L}} &\to \times_{i=1}^m (\mathscr{H}_{\mathscr{L}_i} \cup \{\emptyset\}) \\ B &\mapsto \times_{i=1}^m (B \cap \mathscr{X}_i), \end{aligned}$$

is a bijection. As a consequence, $|\mathscr{H}_{\mathscr{L}}| = \prod_{i=1}^m (|\mathscr{H}_{\mathscr{L}_i}| + 1)$.



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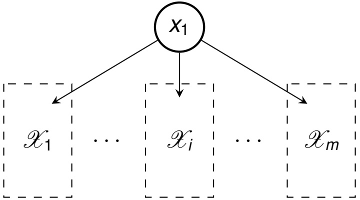

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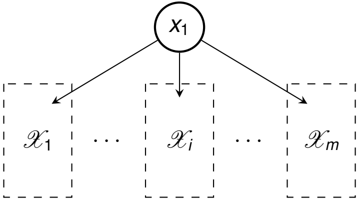

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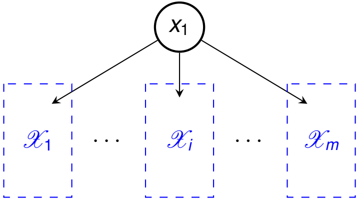

Out-arboresences

Theorem

Let $\mathscr{L}$ be an out-arborescence with root x_1, and let $\{\mathscr{X}_i\}_{i=1}^m$ be the connected components of $\mathscr{L}_{x_1}$. Denote by $\mathscr{L}_i$ the restriction of $\mathscr{L}$ to $\mathscr{X}_i$. Then the following mapping f

$$\begin{aligned} f: Ext(\mathscr{P}(\mathscr{L})) &\to \times_{i=1}^m (Ext(\mathscr{P}(\mathscr{L}_i)) \cup \{\emptyset\}) \\ B &\mapsto \times_{i=1}^m (B \cap \mathscr{X}_i), \end{aligned}$$

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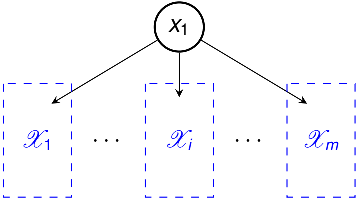
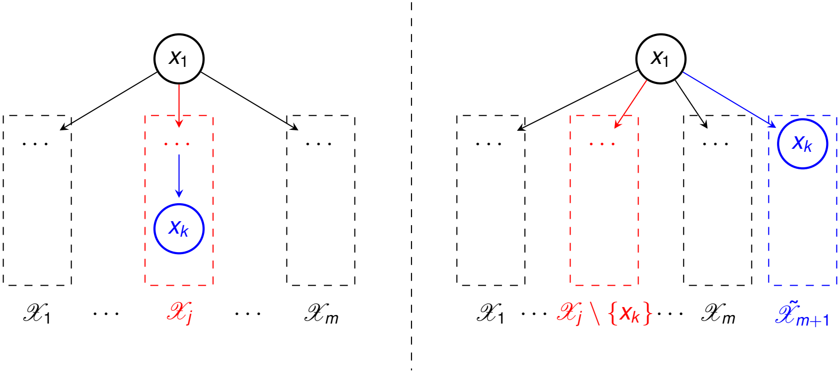


Out-arborescences



Out-arborescences and bounded-above relations

Lemma

Let $\mathscr{L}$ be an out-arborescence over $\mathscr{X}$ with root x_1, and let $\{\mathscr{X}_i\}_{i=1}^m$ be the connected components of $\mathscr{L}_{x_1}$. If there exists some $\mathscr{X}_j$ that is not a singleton, there exists an out-arborescence $\tilde{\mathscr{L}}$ such that $|Ext(\mathscr{P}(\mathscr{L}))| \leq |Ext(\mathscr{P}(\tilde{\mathscr{L}}))|$.

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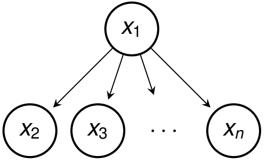
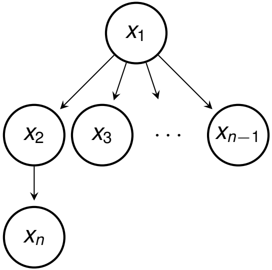


$\mathrm{Ext}(\mathscr{P}(\mathscr{L})) = 2^{n-1}$



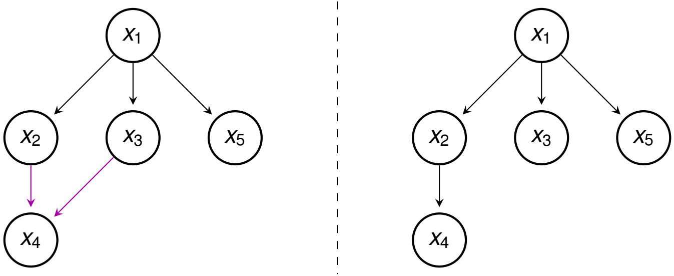
$\mathrm{Ext}(\mathscr{P}(\mathscr{L})) = 3 \cdot 2^{n-3}$

Maximum out-arborescences

Proposition

For any bounded above $\mathscr{L}$ there exists an out-arborescence $\tilde{\mathscr{L}}$ that satisfies $\tilde{\mathscr{L}} \subseteq \mathscr{L}$. Moreover, it holds that $Ext(\mathscr{P}(\mathscr{L})) \subseteq Ext(\mathscr{P}(\tilde{\mathscr{L}}))$, and consequently

$$|Ext(\mathscr{P}(\mathscr{L}))| \leq |Ext(\mathscr{P}(\tilde{\mathscr{L}}))|.$$



Bounded-below relations

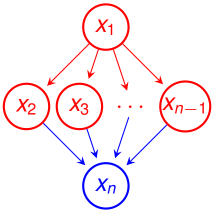
Lemma

Let $\mathscr{L}$ be a bounded-below relation with minimal element x_n. Then

$$Ext(\mathscr{P}(\mathscr{L})) = Ext(\mathscr{P}(\mathscr{L}_{x_n})) \cup \{\mathscr{X}\}, \quad \text{and} \quad |Ext(\mathscr{P}(\mathscr{L}))| = |Ext(\mathscr{P}(\mathscr{L}_{x_n}))| + 1.$$

Corollary

The number of extreme points of $\mathscr{P}(\mathscr{L})$ for a non antimodal bounded-below relation $\mathscr{L}$ is at most $2^{|\mathscr{X}|-2} + 1$.



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Moreover, if $\{\mathscr{X}_i\}_{i=1}^m$ are the connected components of $\mathscr{L}_{x_n}$, then

$$Ext(\mathscr{P}(\mathscr{L})) = \cup_{i=1}^m Ext(\mathscr{P}(\mathscr{L}_i)) \cup \{\mathscr{X}\},$$

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Theorem

The extreme points of $\mathscr{P}(\mathscr{L})$ for an in-arborescence $\mathscr{L}$ are those generated by singletons of $\mathscr{X}$, and $|Ext(\mathscr{P}(\mathscr{L}))| = |\mathscr{X}|$.

Bounded relations

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Theorem

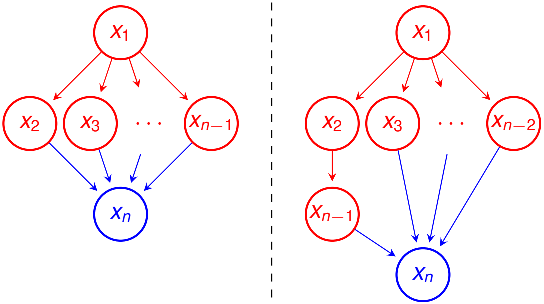
The number of extreme points of $\mathscr{P}(\mathscr{L})$ of a bounded-above relation $\mathscr{L}$ is $2^{|\mathscr{X}|-1}$ if and only if the relation is modal. If the relation $\mathscr{L}$ is non modal, the maximum number of extreme points of $\mathscr{P}(\mathscr{L})$ is $3 \cdot 2^{|\mathscr{X}|-3}$.

Bounded relations

Corollary

Consider a bounded relation $\mathscr{L}$ with minimal element x_n.

1. *If $\mathscr{L}_{x_n}$ is modal, the number of extreme points of $\mathscr{P}(\mathscr{L})$ is $2^{|\mathscr{X}|-2}+1$.*
2. *If $\mathscr{L}_{x_n}$ is non modal, the number of extreme points of $\mathscr{P}(\mathscr{L})$ is at most $3 \cdot 2^{|\mathscr{X}|-4}+1$.*



Lower probability

- **Coherent lower probability:**

$$\underline{P}(A) = \min\{P(A) : P \in \mathscr{P}(\mathscr{L})\}, \qquad \forall A \subseteq \mathscr{X}.$$

- **Möbius inverse:**

$$m_{\underline{P}}(A) = \sum_{C \subseteq A} (-1)^{|A \setminus C|} \underline{P}(C), \qquad \forall A \subseteq \mathscr{X}.$$

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Lower probability

- **Coherent lower probability:**

$$\underline{P}(A) = \min\{P(A) : P \in \mathrm{Ext}(\mathscr{P}(\mathscr{L}))\}, \qquad \forall A \subseteq \mathscr{X}.$$

- **Möbius inverse:**

$$m_{\underline{P}}(A) = \sum_{C \subseteq A} (-1)^{|A \setminus C|} \underline{P}(C), \qquad \forall A \subseteq \mathscr{X}.$$

Properties of the lower probability

- **2-monotonicity:**

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B), \quad \forall A, B \subseteq \mathscr{X}.$$

- **k-monotonicity:**

$$\underline{P}(\cup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,k\}} (-1)^{|I|+1} \underline{P}(\cap_{i \in I} A_i), \quad \forall A_1, \dots, A_k \subseteq \mathscr{X}.$$

- **Belief function:** k-monotone for all $k \in \mathbb{N}$.

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- **Belief function:** k-monotone for all $k \in \mathbb{N}$

$$m_{\underline{P}}(A) = \sum_{C \subseteq A} (-1)^{|A \setminus C|} \underline{P}(C) \geq \mathbf{0}, \quad \forall A \subseteq \mathscr{X}.$$

2-monotonicity

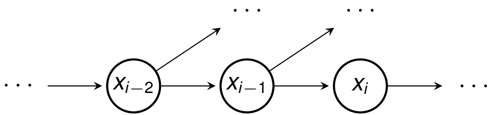
Proposition

Consider a relation $\mathscr{L}$ over $\mathscr{X} = \{x_1, \ldots, x_n\}$, such that there exist $x_i, x_{i-1}, x_{i-2} \in \mathscr{X}$ that verify

$$|\langle x_{i-2}\rangle| \geq 2, \;\; \langle x_{i-1}\rangle = \langle x_{i-2}\rangle \cup \{x_{i-1}\}, \;\; \langle x_i\rangle = \langle x_{i-2}\rangle \cup \{x_{i-1}, x_i\},$$

where $\langle x_i\rangle := \{x_j \in \mathscr{X} : x_j \succeq x_i\}$ is the set of ancestors of x_i.

Then the lower probability associated with $\mathscr{L}$ is not 2-monotone.



Belief functions

Theorem

Let $\mathscr{L}$ be a comparative probability ordering on $\mathscr{X}$ and let $\underline{P}$ be its lower probability.

1. *If $\mathscr{L}$ is modal with maximal element x_1, then $\underline{P}$ is given by*

$$\underline{P}(A) = \begin{cases} 0, & \text{if } x_1 \notin A \\ \dfrac{1}{n - |B|}, & \text{if } A = B \cup \{x_1\}. \end{cases}$$

2. *If $\mathscr{L}$ is antimodal with minimal element x_n, then $\underline{P}$ is given by*

$$\underline{P}(A) = \begin{cases} 0, & \text{if } \exists x_i \in \mathscr{X} \setminus A \quad x_i \neq x_n \\ \dfrac{n-1}{n}, & \text{if } A = \{x_1, \dots, x_{n-1}\} \\ 1, & \text{if } A = \mathscr{X}. \end{cases}$$

In both cases, $\underline{P}$ is a belief function.

Conclusions

Behaviour of the extreme points

Sharper bounds in the cases of

- Bounded-above relations
- Bounded-below relations
- Bounded relations

Explicit expressions and **computational methods** in the cases of

- Modal and antimodal relations
- In- and out-arborescences

Properties of the lower probability

- Many situations where 2-monotonicity **does not hold**
- Belief functions: **modal** and **antimodal** relations

Future work

To be continued...

- To exhaustively **characterize 2-monotonicity**.
- To explore the connection between a **relation and its transpose**.
- To tackle **more general scenarios**: comparative orderings on events, infinite possibility spaces, graded versions of ratios...

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An order-theoretic study of the credal sets generated by comparative probabilities

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