The Imprecise Kolmogorov Model for robustifying p-boxes

David Nieto-Barba, Ignacio Montes, Enrique Miranda

University of Oviedo

UNIVERSIDAD
OVIEDO

SMPS 2026, Lecce

Summary

- We consider a distortion procedure for lower probabilities that transforms p-boxes into p-boxes.
- Our model is based on the Kolmogorov distance between cumulative distribution functions.
- We investigate the main properties of the model as a distortion procedure.

Our setting

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be a *totally ordered* finite possibility space: $x_i \preceq x_j \iff i \leq j$.

We denote by $\mathbb{P}(\mathcal{X})$ the set of probability measures on $\mathcal{P}(\mathcal{X})$ and by

$$\mathcal{L}(\mathcal{X}) := \{f : \mathcal{X} \to \mathbb{R}\}$$

the set of gambles on $\mathcal{X}$.

Lower probabilities and previsions

A lower probability a monotone increasing function $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ s.t. $\underline{P}(\emptyset) = 0, \underline{P}(\mathcal{X}) = 1$. Its conjugate upper probability $\overline{P}$ is given by $\overline{P}(A) = 1 - \underline{P}(A^c)$.

The associated credal set $\mathcal{M}(\underline{P})$ is given by

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) \mid \overline{P}(A) \geq P(A) \geq \underline{P}(A) \quad \forall A \subseteq \mathcal{X}\}.$$

Similarly, a lower prevision is a function $\underline{P} : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$, and its conjugate upper prevision $\overline{P}$ is given by $\overline{P}(f) = -\underline{P}(-f)$.

The associated credal set $\mathcal{M}(\underline{P})$ is given by

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) \mid \overline{P}(f) \geq P(f) \geq \underline{P}(f) \quad \forall f \in \mathcal{L}(\mathcal{X})\}.$$

Avoiding sure loss and coherence

A lower prevision (resp., probability) $\underline{P}$ avoids sure loss when $\mathcal{M}(\underline{P})$ is non-empty, and it is coherent when it can be recovered as the lower envelope of $\mathcal{M}(\underline{P})$:

$$\underline{P}(f) = \min\{P(f) : P \in \mathcal{M}(\underline{P})\} \quad \forall f \in \mathcal{L}(\mathcal{X})$$

or

$$\underline{P}(A) = \min\{P(A) : P \in \mathcal{M}(\underline{P})\} \quad \forall A \subseteq \mathcal{X}.$$

Coherence will be the minimal rationality requirement in this work.

Particular cases

As particular cases of coherent lower probabilities, we have:

- 2-monotone capacities: $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathcal{X}$.
- Belief functions: $\underline{P}(\cup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \underline{P}(\cap_{i \in I} A_i)$ $\forall A_1, \ldots, A_k \subseteq \mathcal{X}, \quad \forall k \in \mathbb{N}$.
- Minitive measures: $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\} \quad \forall A, B \subseteq \mathcal{X}$.
- p-boxes.

Particular cases

As particular cases of coherent lower probabilities, we have:

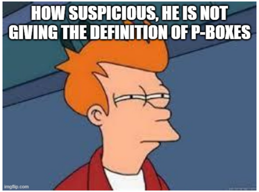
- 2-monotone capacities: $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathcal{X}$.
- Belief functions: $\underline{P}(\cup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \underline{P}(\cap_{i \in I} A_i)$ $\forall A_1, \ldots, A_k \subseteq \mathcal{X}, \quad \forall k \in \mathbb{N}$.
- Minitive measures: $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\} \quad \forall A, B \subseteq \mathcal{X}$.
- p-boxes.

$\hookrightarrow$ The more general model of coherent lower previsions allow in particular to encompass comparative probabilities and other types of credal sets.

Particular cases

As particular cases of coherent lower probabilities, we have:

- 2-monotone capacities: $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathcal{X}$.
- Belief functions: $\underline{P}(\cup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \underline{P}(\cap_{i \in I} A_i)$ $\forall A_1, \ldots, A_k \subseteq \mathcal{X}, \quad \forall k \in \mathbb{N}$.
- Minitive measures: $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\} \quad \forall A, B \subseteq \mathcal{X}$.
- p-boxes.



So what is a p-box?

A p-box is a couple $(\underline{F}, \overline{F})$ of cumulative distribution functions $\underline{F}, \overline{F} : \mathcal{X} \to [0, 1]$ that are monotone with respect to the total order $\preceq$.

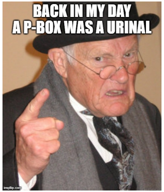
Its associated credal set is

$$\mathcal{M}(\underline{F}, \overline{F}) := \{P \in \mathbb{P}(\mathcal{X}) \mid \underline{F}(x) \leq F_P(x) \leq \overline{F}(x) \quad \forall x \in \mathcal{X}\},$$

where F_P is the CDF associated with P, and $\mathcal{M}(\underline{F}, \overline{F}) \neq \emptyset$ if and only if $\underline{F} \leq \overline{F}$.

The lower envelopes of $\mathcal{M}(\underline{F}, \overline{F})$ determine coherent lower probabilities and previsions.

$\hookrightarrow$ In particular, the associated coherent lower probabilities are belief functions!



From lower probabilities to p-boxes

A lower and an upper probability determine a p-box by

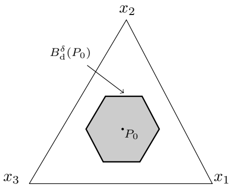
$$\underline{F}_{\underline{P}}(x_i) := \underline{P}(\{x_1, \ldots, x_i\}) \text{ and } \overline{F}_{\overline{P}}(x_i) := \overline{P}(\{x_1, \ldots, x_i\}) \quad \forall i \in \{1, \ldots, n\}.$$

However, in general it holds that $\mathcal{M}(\underline{P}) \subseteq \mathcal{M}(\underline{F}_{\underline{P}}, \overline{F}_{\overline{P}})$, with equality if and only if $\underline{P}$ is determined by a p-box $(\underline{F}, \overline{F})$.

Distortion models

One natural procedure to obtain a coherent lower probability is to consider a distortion model: we consider a neighbourhood around a probability measure P_0 determined some distortion function d and by a given radius δ, which may represent:

- the proportion of contaminated data;
- the fee taken by the betting house;
- the distance to the original model we want to be robust;
- ...



Example: the Kolmogorov model

Given $Q, P \in \mathbb{P}(\mathcal{X})$, their Kolmogorov distance is

$$d_{\mathrm{KM}}(Q, P) := \max_{x \in \mathcal{X}} |F_P(x) - F_Q(x)|.$$

d_{KM} determines the neighbourhood

$$B^{\delta}_{\mathrm{KM}}(P) = \{Q \in \mathbb{P}(\mathcal{X}) \mid F_P(x) - \delta \leq F_Q(x) \leq F_P(x) + \delta \quad \forall x \in \mathcal{X}\}.$$

which coincides with the credal set of a coherent p-box with CDFs

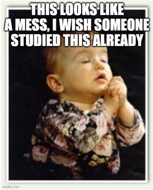
$$\underline{F}(x) := \max\{F_P(x) - \delta, 0\} \quad \text{and} \quad \overline{F}(x) := \min\{F_P(x) + \delta, 1\},$$

for every $x \in \mathcal{X} \setminus \{x_n\}$ and $\underline{F}(x_n) = \overline{F}(x_n) := 1$, meaning that $\mathcal{M}(\underline{F}, \overline{F}) = B^{\delta}_{\mathrm{KM}}(P)$.

So, are we done?

In some cases we may be interested in distorting a model that is already imprecise:

- To transform a lower probability that does not avoid sure loss into one that does;
- To obtain a model with better properties, while keeping the available information;
- To reduce the number of extreme points of the credal set;
- In a consensus seeking process;
- ...



Distortions of lower probabilities

Let $\Lambda = [0,\infty)$ and $\{\phi_\delta : [0,1] \to [0,1]\}_{\delta\in\Lambda}$ be a family of non-decreasing functions such that $\phi_\delta(t) \leq t$ for each $t \in [0,1]$ and $\delta \in \Lambda$.

Given a lower probability $\underline{P}$ and $\delta \in \Lambda$, its distorted lower probability with distorting factor δ is defined as

$$\underline{Q}_\delta[\underline{P}](A) := (\phi_\delta \circ \underline{P})(A) = \phi_\delta\left(\underline{P}(A)\right)$$

for every $A \subset \mathcal{X}$ and $\underline{Q}_\delta[\underline{P}](\mathcal{X}) := 1$.

Example: the Imprecise Total Variation

Let $\underline{P}$ be a coherent lower probability and $\delta > 0$. The imprecise total variation model induced by $(\underline{P}, \delta)$ is the lower probability $\underline{Q}_\delta$ given by

$$\underline{Q}_\delta(A) = \max\left\{\underline{P}(A) - \delta, 0\right\} \quad \forall A \subset \mathcal{X}, \quad \underline{Q}_\delta(\mathcal{X}) = 1.$$

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

(P2) (Semigroup) (a) $\underline{Q}_0[\underline{P}] = \underline{P}$; and (b) $\underline{Q}_{\delta_2+\delta_1}[\underline{P}] = \underline{Q}_{\delta_2}\left[\underline{Q}_{\delta_1}[\underline{P}]\right]$ for any $\delta_1, \delta_2 \in \Lambda$.

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

(P2) (Semigroup) (a) $\underline{Q}_0[\underline{P}] = \underline{P}$; and (b) $\underline{Q}_{\delta_2+\delta_1}[\underline{P}] = \underline{Q}_{\delta_2}\left[\underline{Q}_{\delta_1}[\underline{P}]\right]$ for any $\delta_1, \delta_2 \in \Lambda$.

(P3) (Structure preservation) If $\underline{P}$ belongs to a subfamily $\mathcal{H}$ of lower probabilities, then so does $\underline{Q}_\delta[\underline{P}]$ for every $\delta \in \Lambda$.

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

(P2) (Semigroup) (a) $\underline{Q}_0[\underline{P}] = \underline{P}$; and (b) $\underline{Q}_{\delta_2+\delta_1}[\underline{P}] = \underline{Q}_{\delta_2}\left[\underline{Q}_{\delta_1}[\underline{P}]\right]$ for any $\delta_1, \delta_2 \in \Lambda$.

(P3) (Structure preservation) If $\underline{P}$ belongs to a subfamily $\mathcal{H}$ of lower probabilities, then so does $\underline{Q}_\delta[\underline{P}]$ for every $\delta \in \Lambda$.

(P4) (Reversibility) Given $\delta \in \Lambda$ and $\underline{P}$, there exists ϕ_δ^{-1} such that $\underline{P} = \phi_\delta^{-1} \circ \underline{Q}_\delta[\underline{P}]$.

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

(P2) (Semigroup) (a) $\underline{Q}_0[\underline{P}] = \underline{P}$; and (b) $\underline{Q}_{\delta_2+\delta_1}[\underline{P}] = \underline{Q}_{\delta_2}\left[\underline{Q}_{\delta_1}[\underline{P}]\right]$ for any $\delta_1, \delta_2 \in \Lambda$.

(P3) (Structure preservation) If $\underline{P}$ belongs to a subfamily $\mathcal{H}$ of lower probabilities, then so does $\underline{Q}_\delta[\underline{P}]$ for every $\delta \in \Lambda$.

(P4) (Reversibility) Given $\delta \in \Lambda$ and $\underline{P}$, there exists ϕ_δ^{-1} such that $\underline{P} = \phi_\delta^{-1} \circ \underline{Q}_\delta[\underline{P}]$.

(P5) (Expression as an imprecise neighbourhood model) There exists a *distorting function* $d : \mathbb{P}(\mathcal{X}) \times \underline{\mathbb{P}}(\mathcal{X}) \to [0, \infty)$ such that, given $\underline{P}$, for every $\delta \in \Lambda$ and $Q \in \mathbb{P}(\mathcal{X})$

$$\mathcal{M}(\underline{Q}_\delta[\underline{P}]) = \{Q \in \mathbb{P}(\mathcal{X}) \mid d(Q, \underline{P}) \leq \delta\} =: B_d^\delta(\underline{P}).$$

Desirable properties

(P1) (Expansion) For any $\underline{P}$ and $\delta_1, \delta_2 \in \Lambda$, $\delta_1 \geq \delta_2$ implies $\underline{Q}_{\delta_1}[\underline{P}] \leq \underline{Q}_{\delta_2}[\underline{P}]$.

(P2) (Semigroup) (a) $\underline{Q}_0[\underline{P}] = \underline{P}$; and (b) $\underline{Q}_{\delta_2+\delta_1}[\underline{P}] = \underline{Q}_{\delta_2}\left[\underline{Q}_{\delta_1}[\underline{P}]\right]$ for any $\delta_1, \delta_2 \in \Lambda$.

(P3) (Structure preservation) If $\underline{P}$ belongs to a subfamily $\mathcal{H}$ of lower probabilities, then so does $\underline{Q}_\delta[\underline{P}]$ for every $\delta \in \Lambda$.

(P4) (Reversibility) Given $\delta \in \Lambda$ and $\underline{P}$, there exists ϕ_δ^{-1} such that $\underline{P} = \phi_\delta^{-1} \circ \underline{Q}_\delta[\underline{P}]$.

(P5) (Expression as an imprecise neighbourhood model) There exists a *distorting function* $d : \mathbb{P}(\mathcal{X}) \times \mathbb{P}(\mathcal{X}) \to [0, \infty)$ such that, given $\underline{P}$, for every $\delta \in \Lambda$ and $Q \in \mathbb{P}(\mathcal{X})$

$$\mathcal{M}(\underline{Q}_\delta[\underline{P}]) = \{Q \in \mathbb{P}(\mathcal{X}) \mid d(Q, \underline{P}) \leq \delta\} =: B_d^\delta(\underline{P}).$$

(P6) (Strong commutativity) If $\underline{P}$ is coherent, then for every $\delta \in \Lambda$ it holds:

$$\mathcal{M}(\underline{Q}_\delta[\underline{P}]) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}(\underline{Q}_\delta[P]).$$

The Imprecise Kolmogorov model

Given $\delta \in [0, +\infty)$, the Imprecise Kolmogorov Model (IKM) $(\underline{G}_\delta[\cdot], \overline{G}_\delta[\cdot])$ is given by :

$$\underline{G}_\delta[\underline{F}](x) := \max\left\{\underline{F}(x) - \delta, 0\right\} \quad \text{and} \quad \overline{G}_\delta[\overline{F}](x) := \min\left\{\overline{F}(x) + \delta, 1\right\},$$

for each $x \in \mathcal{X} \setminus \{x_n\}$, with $\underline{G}_\delta[\underline{F}](x_n) = \overline{G}_\delta[\overline{F}](x_n) := 1$.

The Imprecise Kolmogorov model

Given $\delta \in [0, +\infty)$, the Imprecise Kolmogorov Model (IKM) $(\underline{G}_\delta[\cdot], \overline{G}_\delta[\cdot])$ is given by :

$$\underline{G}_\delta[\underline{F}](x) := \max\left\{\underline{F}(x) - \delta, 0\right\} \quad \text{and} \quad \overline{G}_\delta[\overline{F}](x) := \min\left\{\overline{F}(x) + \delta, 1\right\},$$

for each $x \in \mathcal{X} \setminus \{x_n\}$, with $\underline{G}_\delta[\underline{F}](x_n) = \overline{G}_\delta[\overline{F}](x_n) := 1$.

- It follows by definition that this satisfies expansion (P1) and semigroup (P2). ✓

Structure preservation and reversibility

Concerning structure preservation (P3), the IKM always gives a p-box, which is in particular a belief function (hence 2-monotone and coherent).

Structure preservation and reversibility

Concerning structure preservation (P3), the IKM always gives a p-box, which is in particular a belief function (hence 2-monotone and coherent).

- If $(\underline{F}, \overline{F})$ is a minitive p-box, then $(\underline{G_\delta}[\underline{F}], \overline{G_\delta}[\overline{F}])$ is minitive iff either $\underline{F}(x) = 0$ for any $x \neq x_n$ or $\overline{F}(x) = 1$ for any $x \in \mathcal{X}$.

Structure preservation and reversibility

Concerning structure preservation (P3), the IKM always gives a p-box, which is in particular a belief function (hence 2-monotone and coherent).

- If $(\underline{F}, \overline{F})$ is a minitive p-box, then $(\underline{G}_\delta[\underline{F}], \overline{G}_\delta[\overline{F}])$ is minitive iff either $\underline{F}(x) = 0$ for any $x \neq x_n$ or $\overline{F}(x) = 1$ for any $x \in \mathcal{X}$.

With respect to reversibility (P4), given a p-box $(\underline{F}, \overline{F})$ and $\delta \geq 0$, the IKM satisfies (P4) iff $\delta < \min\left\{\underline{F}(x_1), 1 - \overline{F}(x_{n-1})\right\}$.

Expression as a neighbourhood model

Consider $d_{\mathrm{IKM}} : \mathbb{P}(\mathcal{X}) \times \mathbb{F}(\mathcal{X}) \times \mathbb{F}(\mathcal{X}) \to [0, +\infty)$ given by:

$$d_{\mathrm{IKM}}(Q, \underline{F}, \overline{F}) := \max_{x \in \mathcal{X}} \left\{ \max \left\{ \underline{F}(x) - F_Q(x), F_Q(x) - \overline{F}(x) \right\} \right\},$$

for every $Q \in \mathbb{P}(\mathcal{X})$ and $\underline{F}, \overline{F} \in \mathbb{F}(\mathcal{X})$.

- d_{IKM} generalises the Kolmogorov distance.

(P5) $B^{\delta}_{d_{\mathrm{IKM}}}(\underline{F}, \overline{F}) = \mathcal{M}\left(\underline{G}_{\delta}[\underline{F}], \overline{G}_{\delta}[\overline{F}]\right)$. ✓

Strong Commutativity

Let $(\underline{F}, \overline{F})$ be a coherent p-box with associated lower probability $\underline{P}_{(\underline{F},\overline{F})}$.

$$d_{\mathrm{IKM}}(Q, \underline{P}_{(\underline{F},\overline{F})}) = \min_{P \in \mathcal{M}(\underline{P}_{(\underline{F},\overline{F})})} d_{\mathrm{KM}}(Q, P) \quad \forall Q \in \mathbb{P}(\mathcal{X}).$$

$\hookrightarrow$ Thus, IKMs satisfy strong commutativity (P6). ✓

Conclusions and future work

- IKMs satisfy the main properties of interest for an imprecise distortion model.
- While applicable to arbitrary lower probabilities, they only take into account the associated p-box.
- They also differ from the outcome of the Imprecise Total Variation Model.

Conclusions and future work

- IKMs satisfy the main properties of interest for an imprecise distortion model.
- While applicable to arbitrary lower probabilities, they only take into account the associated p-box.
- They also differ from the outcome of the Imprecise Total Variation Model.

Future work:

- Other possible distortions of p-boxes.
- Connections with game theory.

Some references

S. Moral. *Discounting Imprecise Probabilities*. In: The Mathematics of the Uncertain, 685-697. Springer, 2018.

D. Nieto-Barba, I. Montes, E. Miranda. *The Imprecise Total Variation Model and its connections with game theory*. Fuzzy Sets and Systems, 517, 109448, 2025.

D. Nieto-Barba, E. Miranda, and I. Montes. *Distortions of lower probabilities as a tool for avoiding conflict*. Proceedings of ECSQARU 2025, pages 133–147.

M.C.M. Troffaes, E. Miranda, S. Destercke. *On the connection between probability boxes and possibility measures*. Information Sciences 224, 88–108, 2013.

Funding

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

UNIÓN EUROPEA
Fondo Social Europeo
El FSE invierte en tu futuro

AGENCIA ESTATAL DE INVESTIGACIÓN

Project **PID2022-140585NB-I00** funded by MICIU/AEI/10.13039/501100011033 y por FEDER, UE

Thank you (again) for the attention...

...and for the questions!