Measuring incoherence for lower probabilities

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Who are we?

- Aggregation of imprecise probability models.
- Statistical inference with imprecise data.
- Decision making with imprecise information.
- Stochastic orders.

https://imagine-uo.github.io/index.html

No, but *who* are we?


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What this talk is about

- Measuring the internal conflict of a lower probability $\underline{P}$

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- Measuring the internal conflict of a lower probability $\underline{P}$
- ...from the behavioural point of view of avoiding sure loss and coherence

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- ...under an axiomatic approach

What this talk is about

- Measuring the internal conflict of a lower probability $\underline{P}$
- ...from the behavioural point of view of avoiding sure loss and coherence
- ...under an axiomatic approach
- ...and comparing three different possibilities.

What this talk is *not* about

- Measuring the conflict between a number of lower probabilities $\underline{P}_1, \dots, \underline{P}_n$.
- Aggregating imprecise probability models under conflict.
- Measuring conflict within the framework of Dempster-Shafer theory.

The setting

Let Ω be a finite space. A lower probability on Ω is a map $\underline{P} : \mathcal{P}(\Omega) \to [0,1]$ that is monotone ($A \subseteq B \Rightarrow \underline{P}(A) \leq \underline{P}(B)$) and normalised ($\underline{P}(\emptyset) = 0, \underline{P}(\Omega) = 1$).

Its conjugate upper probability $\overline{P}$ is defined by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \Omega$.

The credal set associated with $\underline{P}$ is given by

$$\begin{aligned}\mathcal{M}(\underline{P}) &:= \{P \in \mathbb{P}(\Omega) : P(A) \geq \underline{P}(A), \forall A \subseteq \Omega\} \\ &= \{P \in \mathbb{P}(\Omega) : P(A) \leq \overline{P}(A), \forall A \subseteq \Omega\}.\end{aligned}$$

Epistemic interpretation

We may interpret $\underline{P}$ as a functional giving lower bounds for some precise but unknown probability measure P.

In that case, we say that $\underline{P}$ avoids sure loss when $\mathcal{M}(\underline{P})$ is non-empty, and that it is coherent when

$$\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A), \forall A \subseteq \Omega.$$

The families of lower probabilities avoiding sure loss (resp., coherent) will be denoted by $\underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)$ (resp., $\underline{\mathbb{P}}^{\mathrm{COH}}(\Omega)$).

Natural extension

While not every lower probability $\underline{P}$ that avoids sure loss is necessarily coherent, for any $\underline{P} \in \underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)$ there is always a minimal dominating coherent lower probability: its natural extension $\underline{E}$, given by

$$\underline{E}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A), \forall A \subseteq \Omega.$$

It holds that $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P})$.

Behavioural interpretation

For any event A, the value $\underline{P}(A)$ is understood as the supremum betting rate on A, in the sense that for any $\epsilon > 0$ the gamble that gives us a reward $1 - (\underline{P}(A) - \epsilon)$ if A happens and $-\underline{P}(A) + \epsilon$ if it does not should be a desirable transaction for us.

This interpretation leads to consider two types of inconsistencies for $\underline{P}$:

- Incurring a sure loss, that means that a combination of transactions that are desirable according to $\underline{P}$ makes us subject to a sure loss, no matter the outcome of the experiment.
- Incoherence, that means that the supremum acceptable betting rate for an event can be raised taking into account the desirability assessments implied by $\underline{P}$, and is therefore not tight.

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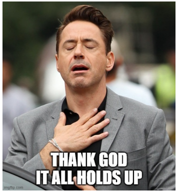
Wait, didn't we define coherence already?

The two notions are equivalent:

- A lower probability incurs a sure loss iff $\mathcal{M}(\underline{P}) = \emptyset$ (i.e., if it does not avoid sure loss).
- It is incoherent iff $\underline{P}(A) \neq \min\{P(A) : P \in \mathcal{M}(\underline{P})\}$ for some A (i.e., if it is not coherent).



So what is the goal of this work?

We consider a lower probability $\underline{P}$ on Ω and want to measure its level of inconsistency, where:

- If $\underline{P}$ is coherent we shall assume that its level of inconsistency is 0.
- In other cases we shall distinguish if it avoids sure loss or not.

There are several ways in which we can do this, and we shall propose a few.

To compare them, we shall consider some desirable properties.

P1- Normalisation

We follow some of the ideas by Bronevich in the context of belief functions.

A first desirable property on a measure of conflict $\mu : \underline{\mathbb{P}}(\Omega) \to \mathbb{R}$ is normalisation: we would like μ to take values in [0,1], so that $\mu(\underline{P}) = 0$ indicates a situation of minimal conflict and $\mu(\underline{P}) = 1$ refers to a situation of maximal conflict.

P2-Monotonicity

We shall also investigate if the measure of conflict μ complies with monotonicity:

$$\underline{P}_1 \leq \underline{P}_2 \Rightarrow \mu(\underline{P}_1) \leq \mu(\underline{P}_2).$$

This means that becoming more imprecise will always reduce the conflict, and in particular that the vacuous model, given by $\underline{P}(A) = 0$ if $A \neq \Omega$ and $\underline{P}(\Omega) = 1$ will give a situation of minimal conflict.

It also means that aggregation by disjunction (=minimum of the lower probabilities) reduces the conflict, while aggregation by conjunction (=natural extension of the maximum) increases it.

P3- Irrelevance

Consider two spaces $\Omega \subseteq \Omega'$, a lower probability $\underline{P}$ on $\mathcal{P}(\Omega)$ and let us define $\underline{P}'$ on $\mathcal{P}(\Omega')$ by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. We say that μ satisfies irrelevance preservation if $\mu(\underline{P}) = \mu(\underline{P}')$.

This means that the measure on a lower probability $\underline{P}$ and on its cylindrical extension to a larger domain is the same.

Some useful property

Under the previous conditions:

$$P' \in \mathcal{M}(\underline{P}') \Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } P'(A) = P(A \cap \Omega), \quad \forall A \subseteq \Omega'.$$

As a consequence:

(i) $\underline{P}$ avoids sure loss if and only if $\underline{P}'$ does. ✓

(ii) $\underline{P}$ is coherent if and only if $\underline{P}'$ is. ✓

(iii) If $\underline{E}, \underline{E}'$ are the natural extensions of $\underline{P}, \underline{P}'$, then $\underline{E}'(A) = \underline{E}(A \cap \Omega)\ \forall A \subseteq \Omega'$. ✓

P4- Expansibility

Let ϕ be a surjective map between two possibility spaces Ω, Ω', $\phi : \Omega \to \Omega'$. Any $\underline{P}$ on Ω induces $\underline{P}'$ on Ω' by $\underline{P}'(A') := \underline{P}(\phi^{-1}(A'))$, for all $A' \subseteq \Omega'$.

We say that μ satisfies expansibility when $\mu(\underline{P}') \leq \mu(\underline{P})$.

In the particular case where ϕ is a bijection, expansibility implies that $\mu(\underline{P}) = \mu(\underline{P}')$.

P5- Convexity

Finally, we shall say that a measure of conflict μ complies with convexity when for any two lower probabilities $\underline{P}_1, \underline{P}_2$ and any $\alpha \in [0, 1]$,

$$\mu\left(\alpha\underline{P}_1 + (1-\alpha)\underline{P}_2\right) \leq \alpha\mu(\underline{P}_1) + (1-\alpha)\mu(\underline{P}_2).$$

In other words, if the aggregation of two lower probabilities (by mixtures) does not increase the overall conflict.

How do we measure inconsistency?

The first idea is to measure the inconsistency as the distance with a consistent model. When $|\Omega| = n$ finite, we may identify any lower probability with an element of $\mathbb{R}^{2^n}$, and use

$$d_p(\underline{P},\underline{Q}) = \left(\sum_{A\subseteq\Omega} |\underline{P}(A) - \underline{Q}(A)|^p\right)^{\frac{1}{p}}.$$

In particular, we may consider:

- $p = 1$: $d_1(\underline{P},\underline{Q}) = \sum_{A\subseteq\Omega} |\underline{P}(A) - \underline{Q}(A)|$.
- $p = 2$: $d_2(\underline{P},\underline{Q}) = \sqrt{\sum_{A\subseteq\Omega}(\underline{P}(A) - \underline{Q}(A))^2}$.
- $p = \infty$: $d_\infty(\underline{P},\underline{Q}) = \max_{A\subseteq\Omega} |\underline{P}(A) - \underline{Q}(A)|$.

First measure: distance to ASL

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$. We define its measure of incurring a sure loss as

$$\mu_{\mathrm{ASL},p}(\underline{P}) = \min_{\underline{Q} \in \underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)} d_p(\underline{P}, \underline{Q}).$$

Since the p-norm is continuous and $\underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)$ is compact, the minimum is attained.

Monotonicity

- Let $\underline{P}$ be a lower probability, and $\underline{Q} \in \underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)$. Given $\underline{Q}' := \min\{\underline{P}, \underline{Q}\}$, it holds that $\underline{Q}' \in \underline{\mathbb{P}}^{\mathrm{ASL}}(\Omega)$ and $d_p(\underline{P}, \underline{Q}') \leq d_p(\underline{P}, \underline{Q})$.
- As a consequence, the lower probability where $\mu_{\mathrm{ASL},p}(\underline{P})$ is attained is always dominated by $\underline{P}$...
- ...and therefore $\mu_{\mathrm{ASL},p}$ satisfies monotonicity. ✓

Normalisation

Monotonicity implies that, given lower probabilities $\underline{P}_1, \ldots, \underline{P}_k$, on $\mathcal{P}(\Omega)$,

$$\mu_{\mathrm{ASL},p}\left(\min_{1\leq i\leq k} \underline{P}_i\right) \leq \min_{1\leq i\leq k} \mu_{\mathrm{ASL},p}(\underline{P}_i) \leq \max_{1\leq i\leq k} \mu_{\mathrm{ASL},p}(\underline{P}_i) \leq \mu_{\mathrm{ASL},p}\left(\max_{1\leq i\leq k} \underline{P}_i\right)$$

and so the maximum value of μ_{ASL} is attained with

$$\underline{P}(A) = \begin{cases} 1 \text{ if } A \neq \emptyset; \\ 0 \text{ if } A = \emptyset, \end{cases}$$

and is given by

$$\mu_{\mathrm{ASL},p}(\underline{P}) = \begin{cases} \sqrt[p]{\sum_{i=1}^{n-1} \binom{n}{i} \cdot (1 - {}^{i}/{}_{n})^p} & \text{if } p \neq \infty; \\ 1 - {}^{1}/{}_{n} & \text{if } p = \infty. \end{cases}$$

Irrelevance

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$, Ω' a superset of Ω and consider the lower probability $\underline{P}' : \mathcal{P}(\Omega') \to [0, 1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then for any $1 \leq p < \infty$ it holds that

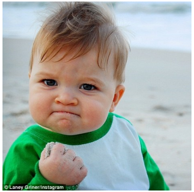
$$\mu_{\mathrm{ASL},p}(\underline{P}') = 2^{\frac{|\Omega' \setminus \Omega|}{p}} \cdot \mu_{\mathrm{ASL},p}(\underline{P}),$$

while $\mu_{\mathrm{ASL},\infty}(\underline{P}') = \mu_{\mathrm{ASL},\infty}(\underline{P})$.

Expansibility

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$ and let $\phi : \Omega \to \Omega'$ be a surjective mapping. Given the lower probability $\underline{P}' := \underline{P} \circ \phi^{-1}$, it holds that $\mu_{\mathrm{ASL},p}(\underline{P}') \leq \mu_{\mathrm{ASL},p}(\underline{P})$.

- Therefore, expansibility holds. ✓



Convexity

Let $\underline{P}_1, \underline{P}_2$ be two lower probabilities on $\mathcal{P}(\Omega)$ and consider $\underline{P} = \alpha\underline{P}_1 + (1-\alpha)\underline{P}_2$ for $\alpha \in (0,1)$. Then $\mu_{\mathrm{ASL},p}(\alpha\underline{P}_1 + (1-\alpha)\underline{P}_2) \leq \alpha\mu_{\mathrm{ASL},p}(\underline{P}_1) + (1-\alpha)\mu_{\mathrm{ASL},p}(\underline{P}_2)$.

- Therefore, convexity holds. ✓

The inequality can be strict, because a convex combination of lower probabilities that incur a sure loss may avoid sure loss. For instance:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$(0.5\underline{P}_1 + 0.5\underline{P}_2)(A)$
{1}	0.4	0.1	0.25
{2}	0.4	0.1	0.25
{3}	0.4	0.1	0.25
{1,2}	0.5	0.7	0.6
{1,3}	0.5	0.7	0.6
{2,3}	0.5	0.7	0.6
{1,2,3}	1	1	1

Second measure: distance to COH

We may instead measure the distance to the closest coherent model:

$$\mu_{\mathrm{COH},p}(\underline{P}) := \min_{\underline{Q} \in \underline{\mathbb{P}}^{\mathrm{COH}}(\Omega)} d_p(\underline{P}, \underline{Q}).$$

A similar idea was proposed by de Bona and Staffel.

Again the properties of the p-norm imply that the minimum is attained.

Normalisation

- $\mu_{\mathrm{COH},p}(\underline{P}) \geq \mu_{\mathrm{ASL},p}(\underline{P})$ for any lower probability $\underline{P}$...

  ... and therefore $\mu_{\mathrm{COH},p}$ does not satisfy normalisation for $1 \leq p < \infty$. ✗

Monotonicity

Since there are incoherent lower probabilities $\underline{P}$ that are bounded above by a probability measure P and on the other hand all of them dominate the vacuous lower probability, we immediately see that it shall not hold. ✗

Even if we restrict our attention to the subclass of incoherent lower probabilities, we cannot assume that the closest coherent model is dominated by the initial lower probability. ✗

Irrelevance

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$, Ω' a superset of Ω and consider the lower probability $\underline{P}' : \mathcal{P}(\Omega') \to [0,1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then for any $1 \leq p < \infty$ it holds that

$$\mu_{\mathrm{COH},p}(\underline{P}') \leq 2^{\frac{|\Omega' \setminus \Omega|}{p}} \cdot \mu_{\mathrm{COH},p}(\underline{P}).$$

Thus, irrelevance only holds for $p = \infty$.

Expansibility

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$ and consider a surjective mapping $\phi : \Omega \to \Omega'$ and the lower probability $\underline{P}' := \underline{P}(\phi^{-1})$. Then $\mu_{\mathrm{COH},p}(\underline{P}) \leq \mu_{\mathrm{COH},p}(\underline{P}')$.

Key to this result is that if a lower probability $\underline{P}$ avoids sure loss (resp., is coherent), so does (resp., is) $\underline{P}'$.

Convexity

Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \to [0, 1]$ be two lower probabilities and consider a convex combination $\underline{P} := \alpha \underline{P}_1 + (1 - \alpha)\underline{P}_2$. Then

$$\mu_{\mathrm{COH},p}(\underline{P}) \leq \alpha \mu_{\mathrm{COH},p}(\underline{P}_1) + (1 - \alpha)\mu_{\mathrm{COH},p}(\underline{P}_2).$$

Thus, convexity holds. ✓

The inequality can be strict, because a convex combination of incoherent lower probabilities may be coherent.

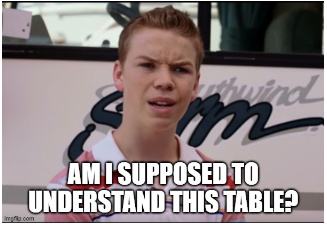
Third approach: distance to the natural extension

When $\underline{P}$ avoids sure loss but is not coherent, we may take its natural extension $\underline{E}$ as the consistent model of reference, and define

$$\mu_{\underline{E},p}(\underline{P}) := d_p(\underline{P}, \underline{E}).$$

- $\mu_{\underline{E},p}(\underline{P}) \geq \mu_{\mathrm{COH},p}(\underline{P})$, but the inequality may be strict:

A	$\underline{P}(A)$	$\underline{E}(A)$	$\underline{Q}(A)$	$\underline{Q}'(A)$
{1}	0.1	0.2	0.1	0.15
{2}	0.2	0.2	0.2	0.25
{3}	0.3	0.4	0.3	0.35
{1,2}	0.5	0.5	0.5	0.45
{1,3}	0.7	0.7	0.6	0.65
{2,3}	0.7	0.7	0.7	0.7
{1,2,3}	1	1	1	1



Normalisation and monotonicity

It holds that

$$\mu_{\underline{E},p}(\underline{P}) \geq \mu_{\mathrm{COH},p}(\underline{P}) \geq \mu_{\mathrm{ASL},p}(\underline{P}),$$

and therefore $\mu_{\underline{E},p}$ may take values greater than 1 if $1 \leq p < \infty$. ✗

On the other hand, $\underline{P}_1 \leq \underline{P}_2$ does not imply that $\mu_{\underline{E},p}(\underline{P}_1) \leq \mu_{\underline{E},p}(\underline{P}_2)$, nor $\mu_{\underline{E},p}(\underline{P}_1) \geq \mu_{\underline{E},p}(\underline{P}_2)$: monotonicity does not hold either. ✗

Irrelevance and expansibility

- Let $\Omega' \supseteq \Omega$ and $\underline{P}' : \mathcal{P}(\Omega') \to [0, 1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then $\mu_{\underline{E}}(\underline{P}') = 2^{\frac{|\Omega' \setminus \Omega|}{p}} \cdot \mu_{\underline{E}}(\underline{P})$.
- Consider a surjective mapping $\phi : \Omega \to \Omega'$ and the lower probability $\underline{P}' := \underline{P}(\phi^{-1})$. If $\underline{P}$ avoids sure loss, then $\mu_{\underline{E}}(\underline{P}') \leq \mu_{\underline{E}}(\underline{P})$.

$\hookrightarrow$ Therefore, irrelevance does not hold ✗ but expansibility does. ✓

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Convexity

Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \to [0,1]$ be two lower probabilities that avoid sure loss and let $\underline{P} := \alpha \underline{P}_1 + (1-\alpha)\underline{P}_2$. Then

$$\mu_{\underline{E},p}(\underline{P}) \leq \alpha \mu_{\underline{E},p}(\underline{P}_1) + (1-\alpha)\mu_{\underline{E},p}(\underline{P}_2).$$

The inequality may be strict, because a convex combination of two lower probabilities avoiding sure loss but not coherent may be coherent:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$\underline{P}(A)$
{1}	$1/6$	$1/3$	0.25
{2}	$1/6$	$1/3$	0.25
{3}	$1/6$	$1/3$	0.25
{1,2}	$2/3$	$1/3$	0.5
{1,3}	$2/3$	$1/3$	0.5
{2,3}	$2/3$	$1/3$	0.5
{1,2,3}	1	1	1

Summary

Property	μ_{ASL}	$\mu_{\mathrm{COH},p}$	$\mu_{\underline{E},p}$
Bounded by 1	✗	✗	✗
Monotone	✓	✗	✗
Irrelevance	✗	✗	✗
Expansibility	✓	✓	✓
Convexity	$\leq$	$\leq$	$\leq$

$\hookrightarrow$ Irrelevance and being bounded by one will only hold in the particular case of $p = \infty$.

Main conclusions

- $\hookrightarrow$ While normalisation does not hold, we can always obtain it because our distances are bounded.
- $\hookrightarrow$ Even if we do not have the equality, we can relate the measure of incoherence in the irrelevance setting.
- $\hookrightarrow$ Becoming more imprecise does not reduce the conflict.

While not presented here, we have also considered two other possibilities:

- $\hookrightarrow$ A measure of inconsistency based on the behavioural interpretation, by Seidenfeld, Schervisch and Kadane.
- $\hookrightarrow$ A measured based on distorting the inconsistent model until we reach a consistent one.

Future work

$\hookrightarrow$ Other reference consistency notions for lower probabilities.

$\hookrightarrow$ Deepen into the connection with distortion models.

$\hookrightarrow$ Other comparison measures between lower probabilities.

$\hookrightarrow$ Internal conflict for lower previsions, and on infinite spaces.

Some references

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Thank you for the attention...

...and for the questions!