Testing convex-ordering dominance via OWA-based skewness coefficients

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University of Oviedo

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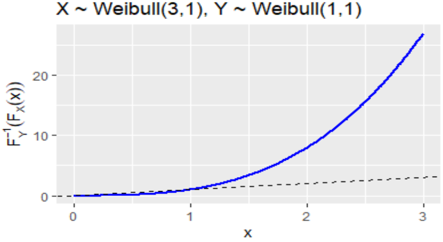
The convex order of van Zwet (1964)

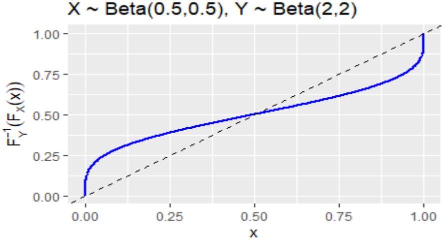
Definition

Let $\mathcal{L}'(\Omega, \mathcal{A}, P)$ be the set of random variables over $(\Omega, \mathcal{A}, P)$ for which the cdf is absolutely continuous, strictly increasing on the preimage of $]0,1[$ and twice differentiable. For any $X, Y \in \mathcal{L}'(\Omega, \mathcal{A}, P)$, we say that X is not more strongly skew to the right than Y, denoted by $X \precsim Y$, if $F_Y^{-1} \circ F_X$ is a convex function on the support of X.

$X \precsim Y$



$X \not\precsim Y$

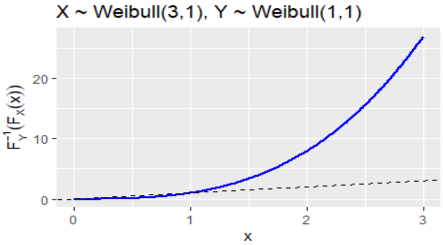


Equivalent definition of the order in terms of concavity

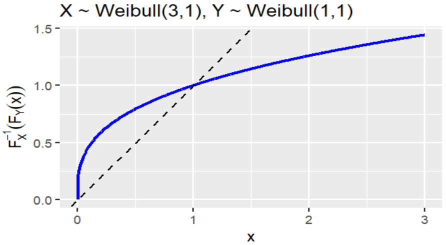
Alternative definition

$F_Y^{-1} \circ F_X$ is a convex function on the support of X if and only if $F_X^{-1} \circ F_Y$ is a concave function on the support of Y. Therefore, saying that $F_Y^{-1} \circ F_X$ is a convex or concave function on the support of X means that $X \precsim Y$ or $Y \precsim X$.

$F_Y^{-1} \circ F_X$ convex



$F_X^{-1} \circ F_Y$ concave



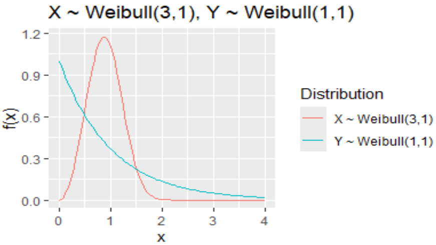
Interpretation of the convex order of van Zwet

Interpretation

X being not more strongly skew to the right than Y intuitively means that Y has a longer tail of extreme values at the right end compared to X.

Skew has no relation with location or scale (it is invariant to translations and scalings).

$X \precsim Y$



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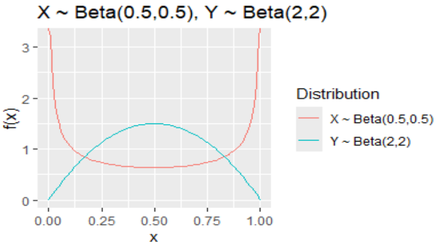
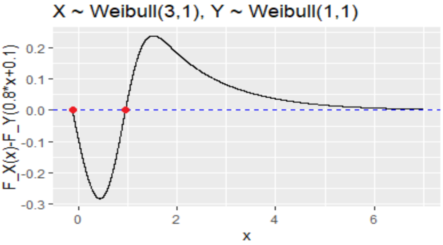


Characterization of the order of van Zwet

Characterization

An equivalent definition of $X \precsim Y$ is that, for any $a, b \in \mathbb{R}$ with $a > 0$, $F_X(x)$ and $F_Y(ax+b)$ cross each other at most twice with the sequence of changes of sign of $F_X(x) - F_Y(ax+b)$ (ignoring zeros) being positive to negative to positive.

$X \precsim Y$



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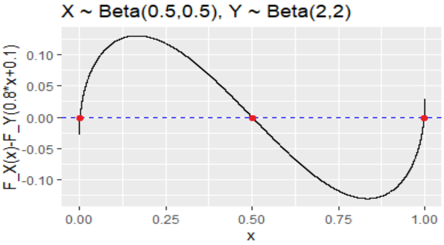
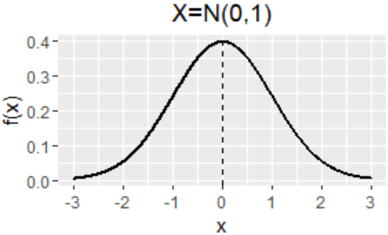


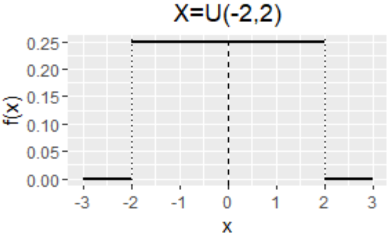
An axiomatization of the notion of skewness (Oja, 1981)

Let $\mathcal{L}(\Omega, \mathcal{A}, P)$ be the set of random variables over $(\Omega, \mathcal{A}, P)$.

A skewness coefficient is a function $\gamma : \mathcal{L}(\Omega, \mathcal{A}, P) \longrightarrow \mathbb{R}$

1. $\gamma(X) = 0$ if X is symmetric.





2. $\gamma(cX + d) = \gamma(X)$ for any $c, d \in \mathbb{R}$ such that $c > 0$.
3. $\gamma(-X) = -\gamma(X)$
4. $\gamma(X) \leq \gamma(Y)$ if $X \precsim Y$.

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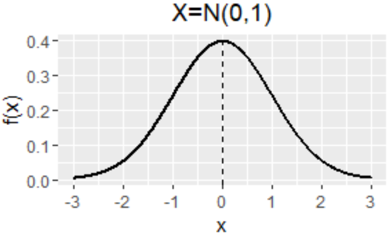

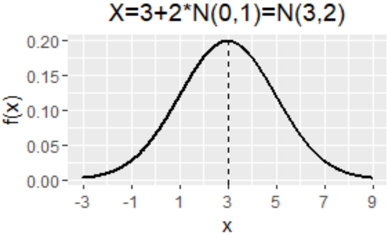

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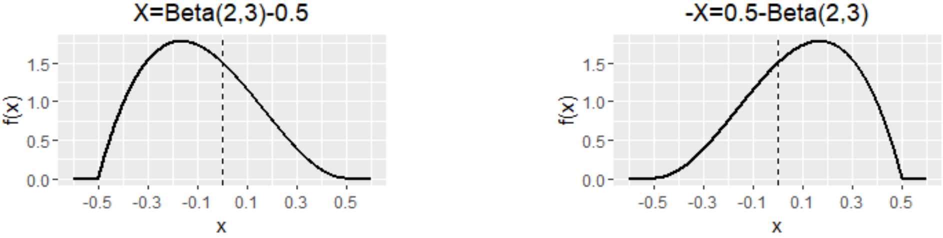

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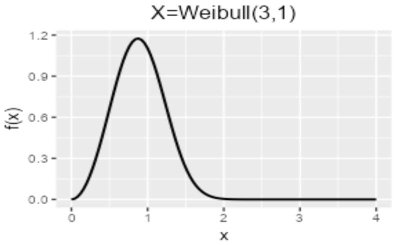

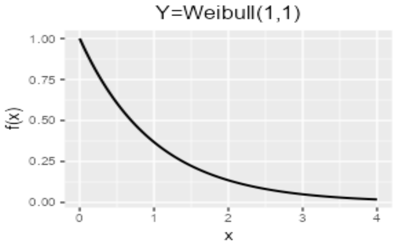

Location-scale-skewness families

Definition

A location-scale-skewness family is a family $\mathcal{F} \subseteq \mathcal{L}'(\Omega, \mathcal{A}, P)$ for which:

- For any $X \in \mathcal{F}$, $c \in \mathbb{R}^+$ and $d \in \mathbb{R}$, it holds that $cX + d \in \mathcal{F}$.
- For any $X, Y \in \mathcal{F}$, it either holds that $F_Y^{-1} \circ F_X$ is a convex or concave function on the support of X.

Examples

In the Weibull family all distributions are comparable in terms of skewness, therefore the shifted-Weibull distribution forms a location-scale-skewness family.

On the contrary, it has been shown that there are parameters of the beta distribution that lead to incomparability in terms of skewness, therefore the beta distribution does not form a location-scale-skewness family.

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A new family of coefficients (Iturrate-Bobes et al., 2025)

Definition of a new family of coefficients

For $X \in \mathcal{L}(\Omega, \mathcal{A}, P)$, we define the family of coefficients:

$$\gamma(X) = \frac{\left((\mathbf{v} - \mathbf{w}) - (\mathbf{w} - \mathbf{v}^r)\right)^T \mathbf{C_p}(X)}{(\mathbf{v} - \mathbf{v}^r)^T \mathbf{C_p}(X)},$$

where

- $\mathbf{C_p}(X) \in \mathbb{R}^m$ are quantiles of X arranged in ascending order such that if $C_q(X) \in \mathbf{C_p}(X)$ for some $q \in]0,1[$, then $C_{1-q}(X) \in \mathbf{C_p}(X)$.
- $\mathbf{v} \in \mathbb{R}^m$ is an asymmetric weigthing vector such that $\sum_{i=1}^{\ell} \mathbf{v}_i \leq \sum_{i=1}^{\ell} \mathbf{v}_i^r$ for all $\ell \in \{1, \ldots, \lfloor m/2 \rfloor\}$.
- $\mathbf{w} \in \mathbb{R}^m$ is a symmetric weighting vector.

Properties of the coefficients (Iturrate-Bobes et al., 2025)

Well-defined

The family of coefficients γ is well-defined for any absolulety continuous random variable.

Oja axioms 1, 2 and 3

The family of coefficients γ fulfills the first three axioms of Oja for any absolulety continuous random variable whose cdf is strictly increasing on the preimage of the interval $]0,1[$.

Oja axiom 4

The family of coefficients γ with m odd and weighting vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v}_i = 0$ for all $i \in \{1, \ldots, \lceil m/2 \rceil\}$ and $\mathbf{w} = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ fulfills the four axioms of Oja for any absolulety continuous random variable whose cdf is strictly increasing on the preimage of the interval $]0,1[$ and twice differentiable (i.e., if we are in $\mathcal{L}'(\Omega, \mathcal{A}, P)$).

Relation with OWAs (Iturrate-Bobes et al., 2025)

Definition

Let $\mathbf{X}$ be a random sample from $X \in \mathcal{L}(\Omega, \mathcal{A}, P)$. The sample version of γ is defined as:

$$\hat{\gamma}(\mathbf{X}) = \frac{\left((\mathbf{v} - \mathbf{w}) - (\mathbf{w} - \mathbf{v}^r)\right)^T \mathbf{X}_{(\lceil n\mathbf{p} \rceil)}}{(\mathbf{v} - \mathbf{v}^r)^T \mathbf{X}_{(\lceil n\mathbf{p} \rceil)}} .$$

A sufficient condition for it to be well-defined (almost surely):

$$X \in \mathcal{L}'(\Omega, \mathcal{A}, P) \qquad \text{and} \qquad n \geq \frac{1}{\min_{i=1,\ldots,m} \mathbf{p}_{i+1} - \mathbf{p}_i} .$$

Asymptotic distribution (Iturrate-Bobes et al., 2025)

Proposition

Let **X** be a random sample from $X \in \mathcal{L}'(\Omega, \mathcal{A}, P)$. It holds that

$$\sqrt{n}\big(\hat{\gamma}(\mathbf{X}) - \gamma(X)\big) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau),$$

with

$$\tau^2 = \frac{4}{\left[(\mathbf{v}^T - \mathbf{v}^{r^T})\mathbf{C}_{\mathbf{p}}(X)\right]^4} \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\min\{\mathbf{p}_i, \mathbf{p}_j\}(1 - \max\{\mathbf{p}_i, \mathbf{p}_j\})}{f(\mathbf{C}_{\mathbf{p}_i}(X)) f(\mathbf{C}_{\mathbf{p}_j}(X))}$$
$$\cdot \left[\mathbf{v}_i(\mathbf{w}^T - \mathbf{v}^{r^T}) + \mathbf{v}_i^r(\mathbf{v}^T - \mathbf{w}^T) + \mathbf{w}_i(\mathbf{v}^{r^T} - \mathbf{v}^T)\right]\mathbf{C}_{\mathbf{p}}(X)$$
$$\cdot \left[\mathbf{v}_j(\mathbf{w}^T - \mathbf{v}^{r^T}) + \mathbf{v}_j^r(\mathbf{v}^T - \mathbf{w}^T) + \mathbf{w}_j(\mathbf{v}^{r^T} - \mathbf{v}^T)\right]\mathbf{C}_{\mathbf{p}}(X).$$

PROBLEM: τ is unknown in practice.

Estimating τ^2 (Iturrate-Bobes et al., 2025)

Theorem

Let $\mathbf{X}$ be a random sample from $X \in \mathcal{L}'(\Omega, \mathcal{A}, P)$. It holds that

$$\sqrt{n}\frac{\hat{\gamma}(\mathbf{X}) - \gamma(X)}{\hat{\tau}(\mathbf{X})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,,$$

with

$$\begin{aligned}\hat{\tau}^2(\mathbf{X}) = &\frac{4}{\left[(\mathbf{v}^T - \mathbf{v}^{r^T})X_{(\lceil n\mathbf{p}\rceil)}\right]^4} \sum_{j=1}^{m}\sum_{i=1}^{m} \frac{\min\{\mathbf{p}_i, \mathbf{p}_j\}(1-\max\{\mathbf{p}_i, \mathbf{p}_j\})}{f_n(\mathbf{X}_{(\lceil n\mathbf{p}\rceil)}) f_n(\mathbf{X}_{(\lceil n\mathbf{p}\rceil)})} \\ &\cdot \left[\mathbf{v}_i(\mathbf{w}^T - \mathbf{v}^{r^T}) + \mathbf{v}_i^r(\mathbf{v}^T - \mathbf{w}^T) + \mathbf{w}_i(\mathbf{v}^{r^T} - \mathbf{v}^T)\right]\mathbf{X}_{(\lceil n\mathbf{p}\rceil)} \\ &\cdot \left[\mathbf{v}_j(\mathbf{w}^T - \mathbf{v}^{r^T}) + \mathbf{v}_j^r(\mathbf{v}^T - \mathbf{w}^T) + \mathbf{w}_j(\mathbf{v}^{r^T} - \mathbf{v}^T)\right]\mathbf{X}_{(\lceil n\mathbf{p}\rceil)}\,,\end{aligned}$$

where f_n is a kernel-based density estimate of the probability density function of X.

Extending the result to two independent samples

Theorem

Let **X** and **Y** be two independent random samples from two independent random variables $X, Y \in \mathcal{L}'(\Omega, \mathcal{A}, P)$. It holds that

$$\frac{(\hat{\gamma}(\mathbf{X}) - \gamma(X)) - (\hat{\gamma}(\mathbf{Y}) - \gamma(Y))}{\sqrt{\frac{\hat{\tau}^2(\mathbf{X})}{n_X} + \frac{\hat{\tau}^2(\mathbf{Y})}{n_Y}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \,,$$

where $\hat{\tau}^2(\mathbf{Z})$, with $\mathbf{Z} = \mathbf{X}$ or $\mathbf{Z} = \mathbf{Y}$, is defined as before.

Remark: We assume $n_X = g(n_Y)$ with $g : \mathbb{N} \to \mathbb{N}$ an increasing function such that $\lim_{n_Y \to \infty} \frac{n_Y}{g(n_Y)} = \lambda \in]0, \infty[$, *i.e.*, both sample sizes grow to infinity at comparable rates.

Tests for convex-ordering

Setting up the hypotheses of the test

The hypotheses to test are the following:

$$H_0 : X \precsim Y \qquad H_1 : X \not\precsim Y\,.$$

A similar problem

Due to property O4 of any skewness coefficient γ, the hypotheses to test could be approximated by:

$$H_0' : \gamma(X) \leq \gamma(Y) \qquad H_1' : \gamma(X) > \gamma(Y)\,.$$

Note

Within location-scale-skewness families that satisfy some specific properties it could be proven that both problems become equivalent. For instance, this is the case within the shifted-Weibull distribution.

Defining the rejection region

Recall that

$$\frac{(\hat{\gamma}(\mathbf{X}) - \gamma(X)) - (\hat{\gamma}(\mathbf{Y}) - \gamma(Y))}{\sqrt{\frac{\hat{\tau}^2(\mathbf{X})}{n_X} + \frac{\hat{\tau}^2(\mathbf{Y})}{n_Y}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \, .$$

Test statistic and rejection region

To perform the test, the following statistic is used with $H_0^* : \gamma(X) = \gamma(Y)$:

$$\frac{\hat{\gamma}(\mathbf{X}) - \hat{\gamma}(\mathbf{Y})}{\sqrt{\frac{\hat{\tau}^2(\mathbf{X})}{n_X} + \frac{\hat{\tau}^2(\mathbf{Y})}{n_Y}}} \xrightarrow[H_0^*]{\mathcal{D}} \mathcal{N}(0,1) \, ,$$

$$\text{R.R.} = \left\{ \mathbf{x} \in \mathbb{R}^{n_X}, \mathbf{y} \in \mathbb{R}^{n_Y} \;\middle|\; \frac{\hat{\gamma}(\mathbf{x}) - \hat{\gamma}(\mathbf{y})}{\sqrt{\frac{\hat{\tau}^2(\mathbf{X})}{n_X} + \frac{\hat{\tau}^2(\mathbf{Y})}{n_Y}}} > z_{1-\alpha} \right\} \, .$$

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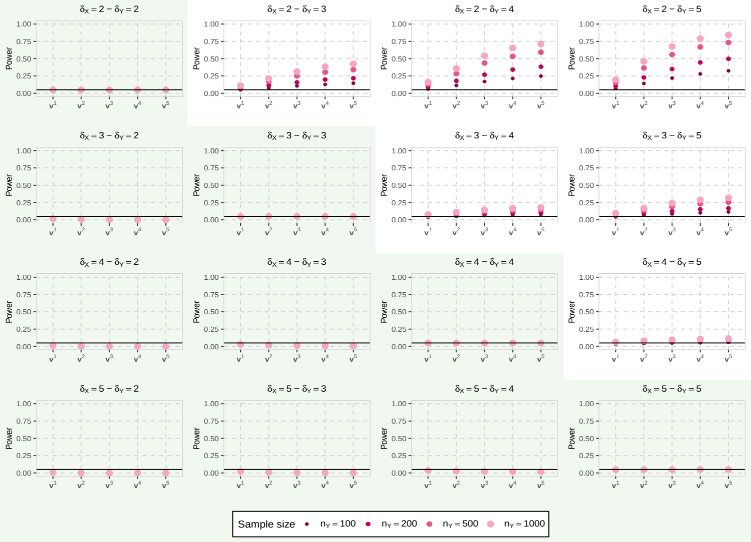
Experiments

Set-up

- Shifted-Weibull distribution with shape parameters $\delta_X, \delta_Y \in \{2, 3, 4, 5\}$ considered.
- Significance level $\alpha = 0.05$.
- The power is estimated through Monte Carlo simulation with 10^5 replications.
- $n_X = 500$ and $n_Y \in \{100, 200, 500, 1000\}$.
- $\mathbf{C_p}$ is such that $\mathbf{p} = (1/6, 2/6, 3/6, 4/6, 5/6)^T$.
- $\mathbf{w} = (0, 0, 1, 0, 0)^T$.
- $\mathbf{v} = \lambda (0, 0, 0, 1, 0)^T + (1 - \lambda) (0, 0, 0, 0, 1)^T$, with $\lambda \in \{0, 1/4, 1/2, 3/4, 1\}$.
- Gaussian kernel and rule-of-thumb bandwidth selector are chosen.

Results


$\delta_X = 2 - \delta_Y = 2$
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$\delta_X = 5 - \delta_Y = 5$
Power
1.00
0.75
0.50
0.25
0.00
v^1 v^2 v^3 v^4 v^5
Sample size $n_Y = 100$ $n_Y = 200$ $n_Y = 500$ $n_Y = 1000$

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Conclusions

Conclusions of the experimental setup

- **Under** H_0: Significance level $\alpha = 0.05$ is preserved for any sample size and weight vector **v**.
- **Under** H_1**:** Large sample sizes are needed for the tests to exhibit high power. The larger the value of λ in the choice of **v** the larger the power.

Future work

The influence of rounding and other sources of imprecision within tests for convex ordering is to be studied in the near future.

IMAGINE

Funding

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