

Measuring incoherence for lower probabilities

Enrique Miranda (with Juan J. Salamanca and Ignacio Montes)



University of Oviedo



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These days, we have seen many different types of imprecise probability models:

- Sets of desirable gambles.
- Credal sets.
- Lower previsions.
- Choice functions.
- ...



These days, we have seen many different types of imprecise probability models:

- Sets of desirable gambles.
- Credal sets.
- Lower previsions.
- Choice functions.
- ...

So many models...I wish there was a place to discuss them all!



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SIPTA

<https://www.ifi.lmu.de/kiml/en/school26/>

<https://www.sipta.org/>

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- Aggregation of imprecise probability models.
- Statistical inference with imprecise data.
- Decision making with imprecise information.
- Stochastic orders.

<https://imagine-uo.github.io/index.html>

No, but *who* are we?



N. Carrizosa



M. Iturrate



E. Miranda



I. Montes



D. Nieto



R. Pérez



J. Salamanca



D. Tagarro

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- Measuring the internal conflict of a lower probability $\underline{P}$

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- ▶ Measuring the internal conflict of a lower probability $\underline{P}$
- ▶ ...from the behavioural point of view of avoiding sure loss and coherence

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- ▶ Measuring the internal conflict of a lower probability $\underline{P}$
- ▶ ...from the behavioural point of view of avoiding sure loss and coherence
- ▶ ...under an axiomatic approach

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- ▶ Measuring the internal conflict of a lower probability $\underline{P}$
- ▶ ...from the behavioural point of view of avoiding sure loss and coherence
- ▶ ...under an axiomatic approach
- ▶ ...and comparing five different possibilities.

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- ▶ Measuring the conflict between a number of lower probabilities $\underline{P}_1, \dots, \underline{P}_n$.
- ▶ Aggregating imprecise probability models under conflict.
- ▶ Measuring conflict within the framework of Dempster-Shafer theory.

The setting

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Let Ω be a finite space. A **lower probability** on Ω is a map $\underline{P} : \mathcal{P}(\Omega) \rightarrow [0, 1]$ that is **monotone** ($A \subseteq B \Rightarrow \underline{P}(A) \leq \underline{P}(B)$) and **normalised** ($\underline{P}(\emptyset) = 0, \underline{P}(\Omega) = 1$).

Its conjugate **upper probability** $\overline{P}$ is defined by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \Omega$.

The **credal set** associated with $\underline{P}$ is given by

$$\begin{aligned} \mathcal{M}(\underline{P}) &:= \{P \in \mathbb{P}(\Omega) : P(A) \geq \underline{P}(A), \forall A \subseteq \Omega\} \\ &= \{P \in \mathbb{P}(\Omega) : P(A) \leq \overline{P}(A), \forall A \subseteq \Omega\}. \end{aligned}$$

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We may interpret $\underline{P}$ as a functional giving lower bounds for some precise but unknown probability measure P .

In that case, we say that $\underline{P}$ **avoids sure loss** when $\mathcal{M}(\underline{P})$ is non-empty, and that it is **coherent** when

$$\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A), \forall A \subseteq \Omega.$$

The families of lower probabilities avoiding sure loss (resp., coherent) will be denoted $\underline{\mathbb{P}}^{\text{ASL}}(\Omega)$ (resp., $\underline{\mathbb{P}}^{\text{COH}}(\Omega)$).

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While not every lower probability $\underline{P}$ that avoids sure loss is necessarily coherent, for any $\underline{P} \in \mathbb{P}^{\text{ASL}}(\Omega)$ there is always a minimal dominating coherent lower probability: its **natural extension** $\underline{E}$, given by

$$\underline{E}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A), \forall A \subseteq \Omega.$$

It holds that $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P})$.

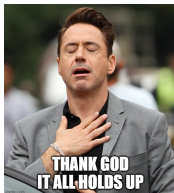
For any event A , the value $\underline{P}(A)$ is understood as the **supremum betting rate** on A , in the sense that for any $\epsilon > 0$ the gamble that gives us a reward $1 - (\underline{P}(A) - \epsilon)$ if A happens and $-\underline{P}(A) + \epsilon$ if it does not should be a desirable transaction for us.

This interpretation leads to consider two types of inconsistencies for $\underline{P}$:

- ▶ **Incurring a sure loss**, that means that a combination of transactions that are desirable according to $\underline{P}$ makes us subject to a sure loss, no matter the outcome of the experiment.
- ▶ **Incoherence**, that means that the supremum acceptable betting rate for an event can be raised taking into account the desirability assessments implied by $\underline{P}$, and is therefore not tight.

The two notions are equivalent:

- ▶ A lower probability incurs a sure loss iff $\mathcal{M}(\underline{P}) = \emptyset$ (i.e., if it does not avoid sure loss).
- ▶ It is incoherent iff $\underline{P}(A) \neq \min\{P(A) : P \in \mathcal{M}(\underline{P})\}$ for some A (i.e., if it is not coherent).



So what is the goal of this work?

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We consider a lower probability $\underline{P}$ on Ω and want to measure its level of inconsistency, where:

- ▶ If $\underline{P}$ is coherent we shall assume that its level of inconsistency is 0.
- ▶ In other cases we shall distinguish if it avoids sure loss or not.

There are several ways in which we can do this, and we shall propose a few.

To compare them, we shall consider some desirable properties.

P1- Normalisation

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We follow some of the ideas by **Bronevich** in the context of belief functions.



A first desirable property on a measure of conflict $\mu : \underline{\mathbb{P}}(\Omega) \rightarrow \mathbb{R}$ is **normalisation**: we would like μ to take values in $[0,1]$, so that $\mu(\underline{P}) = 0$ indicates a situation of minimal conflict and $\mu(\underline{P}) = 1$ refers to a situation of maximal conflict.

We shall also investigate if the measure of conflict μ complies with **monotonicity**:

$$\underline{P}_1 \preceq \underline{P}_2 \Rightarrow \mu(\underline{P}_1) \leq \mu(\underline{P}_2).$$

This means that becoming more imprecise will always reduce the conflict, and in particular that the **vacuous** model, given by $\underline{P}(A) = 0$ if $A \neq \Omega$ and $\underline{P}(\Omega) = 1$ will give a situation of minimal conflict.

It also means that aggregation by **disjunction** (=minimum of the lower probabilities) reduces the conflict, while aggregation by **conjunction** (=natural extension of the maximum) increases it.

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Consider two spaces $\Omega \subseteq \Omega'$, a lower probability $\underline{P}$ on $\mathcal{P}(\Omega)$ and let us define $\underline{P}'$ on $\mathcal{P}(\Omega')$ by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. We say that μ satisfies **irrelevance preservation** if $\mu(\underline{P}) = \mu(\underline{P}')$.

This means that the measure on a lower probability $\underline{P}$ and on its cylindrical extension to a larger domain is the same.

Some useful property

Under the previous conditions:

$$P' \in \mathcal{M}(\underline{P}') \Leftrightarrow \exists P \in \mathcal{M}(\underline{P}) \text{ such that } P'(A) = P(A \cap \Omega), \quad \forall A \subseteq \Omega'.$$

As a consequence:

- (i) $\underline{P}$ avoids sure loss if and only if $\underline{P}'$ does. ✓
- (ii) $\underline{P}$ is coherent if and only if $\underline{P}'$ is. ✓
- (iii) If $\underline{E}, \underline{E}'$ are the natural extensions of $\underline{P}, \underline{P}'$, then $\underline{E}'(A) = \underline{E}(A \cap \Omega) \forall A$. ✓

P4- Expansibility

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Let ϕ be a surjective map between two possibility spaces Ω, Ω' , $\phi : \Omega \rightarrow \Omega'$. Any $\underline{P}$ on Ω induces $\underline{P}'$ on Ω' by $\underline{P}'(A') := \underline{P}(\phi^{-1}(A'))$, for all $A' \subseteq \Omega'$.

We say that μ satisfies **expansibility** when $\mu(\underline{P}') \leq \mu(\underline{P})$.

In the particular case where ϕ is a bijection, expansibility means that $\mu(\underline{P}) = \mu(\underline{P}')$.

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Finally, we shall say that a measure of conflict μ complies with **convexity** when for any two lower probabilities $\underline{P}_1, \underline{P}_2$ and any $\alpha \in [0, 1]$,

$$\mu(\alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2) \leq \alpha \mu(\underline{P}_1) + (1 - \alpha) \mu(\underline{P}_2).$$

In other words, if the aggregation of two lower probabilities (by mixtures) does not increase the overall conflict.

How do we measure inconsistency?

The first idea is to measure the inconsistency as the distance with a consistent model. When $|\Omega| = n$ finite, we may identify any lower probability with an element of $\mathbb{R}^{2^n}$, and use

$$d_p(\underline{P}, \underline{Q}) = \left(\sum_{A \subseteq \Omega} |\underline{P}(A) - \underline{Q}(A)|^p \right)^{\frac{1}{p}}.$$

In particular, we may consider:

$$p = 1: d_1(\underline{P}, \underline{Q}) = \sum_{A \subseteq \Omega} |\underline{P}(A) - \underline{Q}(A)|.$$

$$p = 2: d_2(\underline{P}, \underline{Q}) = \sqrt{\sum_{A \subseteq \Omega} (\underline{P}(A) - \underline{Q}(A))^2}.$$

$$p = \infty: d_\infty(\underline{P}, \underline{Q}) = \max_{A \subseteq \Omega} |\underline{P}(A) - \underline{Q}(A)|.$$

First measure: distance to ASL

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Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$. We define its **measure of incurring a sure loss** as

$$\mu_{\text{ASL},p}(\underline{P}) = \min_{\underline{Q} \in \mathbb{P}^{\text{ASL}}(\Omega)} d_p(\underline{P}, \underline{Q}).$$

Since the p -norm is continuous and $\mathbb{P}^{\text{ASL}}(\Omega)$ is compact, the minimum is attained.

- ▶ Let $\underline{P}$ be a lower probability, and $\underline{Q} \in \mathbb{P}^{\text{ASL}}(\Omega)$. Given $\underline{Q}' := \min\{\underline{P}, \underline{Q}\}$, it holds that $\underline{Q}' \in \mathbb{P}^{\text{ASL}}(\Omega)$ and $d_p(\underline{P}, \underline{Q}') \leq d_p(\underline{P}, \underline{Q})$.
- ▶ As a consequence, the lower probability where $\mu_{\text{ASL},p}(\underline{P})$ is attained is always dominated by $\underline{P}$...
- ▶ ...and therefore $\mu_{\text{ASL},p}$ satisfies monotonicity. ✓

Monotonicity implies that, given lower probabilities $\underline{P}_1, \dots, \underline{P}_k$, on $\mathcal{P}(\Omega)$,

$$\mu_{\text{ASL},p} \left(\min_{i=1,\dots,k} \underline{P}_i \right) \leq \min_{1 \leq i \leq k} \mu_{\text{ASL},p}(\underline{P}_i) \leq \max_{1 \leq i \leq k} \mu_{\text{ASL},p}(\underline{P}_i) \leq \mu_{\text{ASL},p} \left(\max_{i=1,\dots,k} \underline{P}_i \right)$$

and so the maximum value of μ_{ASL} is attained with

$$\underline{P}(A) = \begin{cases} 1 & \text{if } A \neq \emptyset; \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and is given by

$$\mu_{\text{ASL},p}(\underline{P}) = \begin{cases} \sqrt[p]{\sum_{i=1}^{n-1} \binom{n}{i} \cdot (1 - i/n)^p} & \text{if } p \neq \infty; \\ 1 - 1/n & \text{if } p = \infty. \end{cases}$$

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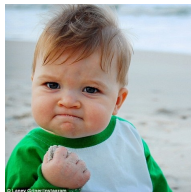
Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$, Ω' a superset of Ω and consider the lower probability $\underline{P}' : \mathcal{P}(\Omega') \rightarrow [0, 1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then for any $1 \leq p < \infty$ it holds that

$$\mu_{\text{ASL},p}(\underline{P}') = 2^{\frac{|\Omega' \setminus \Omega|}{p}} \cdot \mu_{\text{ASL},p}(\underline{P}),$$

while $\mu_{\text{ASL},\infty}(\underline{P}') = \mu_{\text{ASL},\infty}(\underline{P})$.

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$ and let $\phi : \Omega \rightarrow \Omega'$ be a surjective mapping. Given the lower probability $\underline{P}' := \underline{P} \circ \phi^{-1}$, it holds that $\mu_{\text{ASL},p}(\underline{P}') \leq \mu_{\text{ASL},p}(\underline{P})$.

- Therefore, expansibility holds. ✓



Convexity

Let $\underline{P}_1, \underline{P}_2$ be two lower probabilities on $\mathcal{P}(\Omega)$ and consider $\underline{P} = \alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2$ for $\alpha \in (0, 1)$. Then $\mu_{\text{ASL}, p}(\alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2) \leq \alpha \mu_{\text{ASL}, p}(\underline{P}_1) + (1 - \alpha) \mu_{\text{ASL}, p}(\underline{P}_2)$.

► Therefore, convexity holds. ✓

The inequality can be strict, because a convex combination of lower probabilities that incur a sure loss may avoid sure loss. For instance:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$(0.5\underline{P}_1 + 0.5\underline{P}_2)(A)$
{1}	0.4	0.1	0.25
{2}	0.4	0.1	0.25
{3}	0.4	0.1	0.25
{1,2}	0.5	0.7	0.6
{1,3}	0.5	0.7	0.6
{2,3}	0.5	0.7	0.6
{1,2,3}	1	1	1

Second measure: distance to COH

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We may instead measure the **distance to the closest coherent model**:

$$\mu_{\text{COH},p}(\underline{P}) := \min_{\underline{Q} \in \mathbb{P}^{\text{COH}}(\Omega)} d_p(\underline{P}, \underline{Q}).$$

A similar idea was proposed by **de Bona** and **Staffel**.

Again the properties of the p -norm imply that the minimum is attained.

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► $\mu_{\text{COH},p}(\underline{P}) \geq \mu_{\text{ASL},p}(\underline{P})$ for any lower probability $\underline{P}$...

... and therefore $\mu_{\text{COH},p}$ does not satisfy normalisation for $1 \leq p < \infty$. ✗

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Since there are incoherent lower probabilities $\underline{P}$ that are bounded above by a probability measure P and on the other hand all of them dominate the vacuous lower probability, we immediately see that it shall not hold. $\times$

Even if we restrict our attention to the subclass of incoherent lower probabilities, we cannot assume that the closest coherent model is dominated by the initial lower probability. $\times$

Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$, Ω' a superset of Ω and consider the lower probability $\underline{P}' : \mathcal{P}(\Omega') \rightarrow [0, 1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then for any $1 \leq p < \infty$ it holds that

$$\mu_{\text{COH},p}(\underline{P}') \leq 2^{\frac{|\Omega' \setminus \Omega|}{p}} \cdot \mu_{\text{COH},p}(\underline{P}).$$

Thus, irrelevance only holds for $p = \infty$.

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Let $\underline{P}$ be a lower probability on $\mathcal{P}(\Omega)$ and consider a surjective mapping $\phi : \Omega \rightarrow \Omega'$ and the lower probability $\underline{P}' := \underline{P}(\phi^{-1})$. Then $\mu_{\text{COH},p}(\underline{P}) \leq \mu_{\text{COH},p}(\underline{P}')$.

Key to this result is that if a lower probability $\underline{P}$ avoids sure loss (resp., is coherent), so does (resp., is) $\underline{P}'$.

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Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be two lower probabilities and consider a convex combination $\underline{P} := \alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2$. Then

$$\mu_{\text{COH},p}(\underline{P}) \leq \alpha \mu_{\text{COH},p}(\underline{P}_1) + (1 - \alpha) \mu_{\text{COH},p}(\underline{P}_2).$$

Thus, convexity holds. ✓

The inequality can be strict, because a convex combination of incoherent lower probabilities may be coherent.

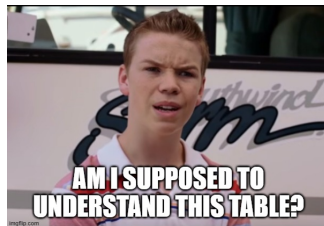
Third approach: distance to the natural extension

When $\underline{P}$ avoids sure loss but is not coherent, we may take its natural extension $\underline{E}$ as the consistent model of reference, and define

$$\mu_{\underline{E},p}(\underline{P}) := d_p(\underline{P}, \underline{E}).$$

► $\mu_{\underline{E},p}(\underline{P}) \geq \mu_{\text{COH},p}(\underline{P})$, but the inequality may be strict:

A	$\underline{P}(A)$	$\underline{E}(A)$	$\underline{Q}(A)$	$\underline{Q}'(A)$
{1}	0.1	0.2	0.1	0.15
{2}	0.2	0.2	0.2	0.25
{3}	0.3	0.4	0.3	0.35
{1,2}	0.5	0.5	0.5	0.45
{1,3}	0.7	0.7	0.6	0.65
{2,3}	0.7	0.7	0.7	0.7
{1,2,3}	1	1	1	1



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It holds that

$$\mu_{\underline{E},p}(\underline{P}) \geq \mu_{\text{COH},p}(\underline{P}) \geq \mu_{\text{ASL},p}(\underline{P}),$$

and therefore $\mu_{\underline{E},p}$ may take values greater than 1 if $1 \leq p < \infty$. **✗**

On the other hand, $\underline{P}_1 \leq \underline{P}_2$ does not imply that $\mu_{\underline{E},p}(\underline{P}_1) \leq \mu_{\underline{E},p}(\underline{P}_2)$, nor $\mu_{\underline{E},p}(\underline{P}_1) \geq \mu_{\underline{E},p}(\underline{P}_2)$: monotonicity does not hold either. **✗**

Irrelevance and expansibility

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► Let $\Omega' \supseteq \Omega$ and $\underline{P}' : \mathcal{P}(\Omega') \rightarrow [0, 1]$ given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$.
Then $\mu_{\underline{E}}(\underline{P}') = 2^{\frac{|\Omega' \setminus \Omega|}{P}} \cdot \mu_{\underline{E}}(\underline{P})$.

► Consider a surjective mapping $\phi : \Omega \rightarrow \Omega'$ and the lower probability $\underline{P}' := \underline{P}(\phi^{-1})$.
If $\underline{P}$ avoids sure loss, then $\mu_{\underline{E}}(\underline{P}') \leq \mu_{\underline{E}}(\underline{P})$.

↪ Therefore, irrelevance does not hold ✗ but expansibility does. ✓

Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be two lower probabilities that avoid sure loss and let $\underline{P} := \alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2$. Then

$$\mu_{\underline{E}, p}(\underline{P}) \leq \alpha \mu_{\underline{E}, p}(\underline{P}_1) + (1 - \alpha) \mu_{\underline{E}, p}(\underline{P}_2).$$

The inequality may be strict, because a convex combination of two lower probabilities avoiding sure loss but not coherent may be coherent:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$\underline{P}(A)$
$\{1\}$	$1/6$	$1/3$	0.25
$\{2\}$	$1/6$	$1/3$	0.25
$\{3\}$	$1/6$	$1/3$	0.25
$\{1,2\}$	$2/3$	$1/3$	0.5
$\{1,3\}$	$2/3$	$1/3$	0.5
$\{2,3\}$	$2/3$	$1/3$	0.5
$\{1,2,3\}$	1	1	1

Fourth approach: SSK

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The approach by **Schervisch, Seidenfeld and Kadane** gives, in the case of a lower probability $\underline{P} : \mathcal{P}(\Omega) \rightarrow [0, 1]$, the value

$$\mu(\underline{P}) = \sup \left\{ \frac{-\min\{0, \sup \sum_{A \subseteq \Omega} \lambda_A (I_A - \underline{P}(A))\}}{f_n(\{\lambda_A (I_A - \underline{P}(A)) : A \subseteq \Omega\})} \right\},$$

where the first supremum is taken over the set of all possible weights $\lambda_A \geq 0$ on the events $A \subseteq \Omega$ and f_n is a **normalisation function** satisfying a number of desirable properties.

Connection with the behavioural interpretation

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If a lower probability incurs a sure loss, this means that there are non-negative weights λ_A such that

$$\sup_{A \subseteq \Omega} \sum \lambda_A (I_A - \underline{P}(A)) < 0.$$

Thus, if we give the values $\underline{P}(A)$ as supremum acceptable betting rates, they could be exploited to build a **Dutch book** against us.

The extent of the loss could be arbitrarily large by choosing wisely the coefficients λ_A , so measuring the loss as the rate of incoherence should be normalised somehow.

What are normalisation functions?

There are several possible normalisation conditions. SSK consider three different approaches:

- ▶ The bookie's escrow, where we consider the most the bookie can lose on a single gamble.
- ▶ The gambler's escrow, which is the most the gambler can lose on a single bet.
- ▶ A third, neutral approach between the bookie and the gambler.

The function f_n is defined on one of these three, and it should satisfy some desirable properties, such as [homogeneity](#) and [continuity](#).

Case under study

We focus on the particular case of $f_n(\{\lambda_A(I_A - \underline{P}(A)) : A \subseteq \Omega\}) = \sum_A \lambda_A$. Then

$$\mu_{\text{SSK}}(\underline{P}) = \sup_{\vec{\lambda} \in \mathcal{H}_\Omega} \left\{ -\min\{0, \max \sum_{A \subseteq \Omega} \lambda_A(I_A - \underline{P}(A))\} \right\},$$

where

$$\mathcal{H}_\Omega := \left\{ \vec{\lambda} \in (\mathbb{R}^+)^{2^\Omega} : \sum_A \lambda_A = 1 \right\}.$$

► $\mu_{\text{SSK}}(\underline{P}) = 0$ if $\underline{P}$ avoids sure loss, and otherwise

$$\mu_{\text{SSK}}(\underline{P}) = \sup_{\vec{\lambda} \in \mathcal{H}} \left\{ -\max \left(\sum_{A \subseteq \Omega} \lambda_A(I_A - \underline{P}(A)) \right) \right\}.$$

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- ▶ As already observed by SSK, $\underline{P}_1 \leq \underline{P}_2 \Rightarrow \mu_{\text{SSK}}(\underline{P}_1) \leq \mu_{\text{SSK}}(\underline{P}_2)$. ✓
- ▶ It can also be checked that $\mu_{\text{SSK}}(\underline{P}) \leq 1$: the worst case scenario is attained with $\underline{P}(A) = 1$ for $A \neq \emptyset$ and constant weights λ_A . ✓

Irrelevance and expansibility

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► Let $\Omega' \supseteq \Omega$ and $\underline{P}' : \mathcal{P}(\Omega') \rightarrow [0, 1]$ be given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$.
Then $\mu_{\text{SSK}}(\underline{P}) = \mu_{\text{SSK}}(\underline{P}')$.

► Consider a surjective mapping $\phi : \Omega \rightarrow \Omega'$ and the lower probability $\underline{P}' := \underline{P}(\phi^{-1})$.
Then $\mu_{\text{SSK}}(\underline{P}) \geq \mu_{\text{SSK}}(\underline{P}')$.

↪ Thus, irrelevance and expansibility hold. ✓

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Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be two lower probabilities, $\alpha \in [0, 1]$ and $\underline{P} := \alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2$.
Then:

$$\mu_{\text{SSK}}(\underline{P}) \leq \alpha \mu_{\text{SSK}}(\underline{P}_1) + (1 - \alpha) \mu_{\text{SSK}}(\underline{P}_2),$$

i.e., μ_{SSK} satisfies convexity. ✓

Fifth approach: distortion models

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Our last approach is to distort the lower probability until it avoids sure loss, and then measure the conflict in the distorted model.

If we let

$$\delta^* = \min\{\delta : \underline{Q}_\delta \text{ avoids sure loss}\}$$

our measure of incoherence would be

$$\mu_{\text{DIST}}(\underline{P}) = \delta^* + \mu_{\underline{E}}(\underline{Q}_{\delta^*}).$$

How do we distort a lower probability?

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Specifically, we would define, for any $\delta > 0$, the lower probability

$$\underline{Q}_\delta(A) = \begin{cases} \max\{\underline{P}(A) - \delta, 0\}, & \text{if } A \neq \Omega, \\ 1, & \text{if } A = \Omega, \end{cases}$$

and then use also the supremum norm when measuring the distance with respect to the natural extension.

If we consider $\underline{P}$ given by $\underline{P}(A) = 1$ for any $A \neq \emptyset$, we obtain $\delta^* = \frac{n-1}{n}$, and

$$\underline{Q}_{\delta^*}(A) = \begin{cases} \frac{1}{n} & \text{if } A \neq \emptyset, \Omega \\ 0 & \text{if } A = \emptyset \\ 1 & \text{if } A = \Omega, \end{cases}$$

which leads to $\mu_{\text{DIST}}(\underline{P}) = \frac{2n-3}{n}$.

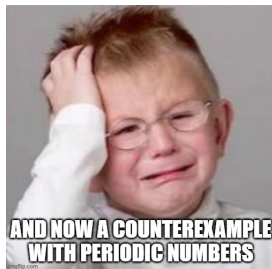
Thus, normalisation does not hold. ✗

Monotonicity

► If $\underline{P}_1 \leq \underline{P}_2$, then $\min \{ \delta : \underline{Q}_{1,\delta} \text{ avoids sure loss} \} \leq \min \{ \delta : \underline{Q}_{2,\delta} \text{ avoids sure loss} \}$. ✓

However, it may be that $\mu_{\text{DIST}}(\underline{P}_1) > \mu_{\text{DIST}}(\underline{P}_2)$:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$\underline{Q}_{1, \frac{0.35}{3}}(A)$	$\underline{Q}_{2, \frac{0.35}{3}}(A)$
{1}	0.25	0.38 $\widehat{3}$	0.1 $\widehat{3}$	0.2 $\widehat{6}$
{2}	0.35	0.38 $\widehat{3}$	0.2 $\widehat{3}$	0.2 $\widehat{6}$
{3}	0.45	0.58 $\widehat{3}$	0. $\widehat{3}$	0.4 $\widehat{6}$
{1,2}	0.65	0.65	0.5 $\widehat{3}$	0.5 $\widehat{3}$
{1,3}	0.85	0.85	0.7 $\widehat{3}$	0.7 $\widehat{3}$
{2,3}	0.85	0.85	0.7 $\widehat{3}$	0.7 $\widehat{3}$
{1,2,3}	1	1	1	1



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► Let Ω' a superset of Ω and $\underline{P}' : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be given by $\underline{P}'(A) = \underline{P}(A \cap \Omega)$ for every $A \subseteq \Omega'$. Then $\mu_{\text{DIST}}(\underline{P}) = \mu_{\text{DIST}}(\underline{P}')$.

↪ Thus, irrelevance holds. ✓

The key here is that $\underline{Q}'_{\delta}$ avoids sure loss if and only if $\underline{Q}_{\delta}$ does, and as a consequence $\delta_*(\underline{P}') = \delta_*(\underline{P})$.

Expansibility

Concerning expansibility, $\delta^*(\underline{P}) \geq \delta^*(\underline{P}')$, since $\underline{P}'$ will avoid sure loss as soon as $\underline{P}$ does. But this does not mean that $\mu_{\text{DIST}}(\underline{P}) \geq \mu_{\text{DIST}}(\underline{P}')$.

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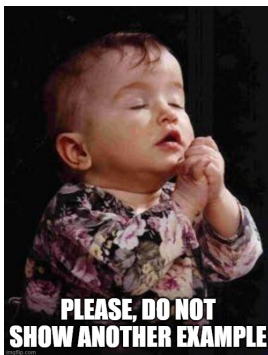
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Expansibility

Concerning expansibility, $\delta^*(\underline{P}) \geq \delta^*(\underline{P}')$, since $\underline{P}'$ will avoid sure loss as soon as $\underline{P}$ does. But this does not mean that $\mu_{\text{DIST}}(\underline{P}) \geq \mu_{\text{DIST}}(\underline{P}')$.

Consider $\Omega = \{1, 2, 3, 4, 5\}$ and the lower probability $\underline{P}$ on $\mathcal{P}(\Omega)$ given by

$$\underline{P}(A) = \begin{cases} 0.15 & \text{if } A = \{5\} \\ 0.3 & \text{if } A = \{1\}, \{2\}, \{3\}, \{4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\} \\ 0.45 & \text{if } A = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 5\}, \\ & \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\} \\ 0.6 & \text{if } A = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}, \\ & \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\} \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mu_{\text{DIST}}(\underline{P}) = 0.15$, but if we consider $\Omega' = \{1, 2, 3, 4\}$ and the surjective mapping $\phi(1) = 1, \phi(2) = 2, \phi(3) = 3, \phi(4) = \phi(5) = 4$, we obtain $\mu_{\text{DIST}}(\underline{P}') = 0.25$.

Convexity

Let $\underline{P}_1, \underline{P}_2 : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be two lower probabilities that incur a sure loss and let $\underline{P} := \alpha \underline{P}_1 + (1 - \alpha) \underline{P}_2$.

Then $\mu_{\text{DIST}}(\underline{P})$ may be smaller but it may also be greater than $\alpha \mu_{\text{DIST}}(\underline{P}_1) + (1 - \alpha) \mu_{\text{DIST}}(\underline{P}_2)$, so convexity does not hold. ✗.

A	$\underline{P}'_1(A)$	$\underline{P}'_2(A)$	$\underline{P}'(A)$
{1}	0.1	0.4	0.25
{2}	0.1	0.5	0.3
{3}	0.6	0.7	0.65
{1,2}	0.8	0.7	0.75
{1,3}	0.6	0.9	0.75
{2,3}	0.6	1	0.8
{1,2,3}	1	1	1

Then $\mu_{\text{DIST}}(\underline{P}_1) = 0.2 = \mu_{\text{DIST}}(\underline{P}_2), \mu_{\text{DIST}}(\underline{P}) = 0.25$.

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Property	μ_{ASL}	$\mu_{\text{COH},p}$	$\mu_{\underline{E},p}$	μ_{SSK}	μ_{DIST}
Bounded by 1	✗	✗	✗	✓	✗
Monotone	✓	✗	✗	✓	✗
Irrelevance	✗	✗	✗	✓	✓
Expansibility	✓	✓	✓	✓	✗
Convexity	$\leq$	$\leq$	$\leq$	$\leq$	✗

Main conclusions

- ↪ Teddy's measure seems to perform better overall.
- ↪ While normalisation does not hold, we can always obtain it because our distances are bounded.
- ↪ Even if we do not have the equality, we can relate the measure of incoherence in the irrelevance setting.
- ↪ Becoming more imprecise does not reduce the conflict.



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- ↪ Study of SSK approach with other normalisation conditions.
- ↪ Other distortion models as a measure of conflict.
- ↪ Other comparison measures between lower probabilities.
- ↪ Internal conflict for lower previsions, and on infinite spaces.

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Thank you for the attention...

...and for the questions!

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